1.4 Compound Inequalities

This section discusses a technique that is used to solve compound inequalities, which is a phrase that usually refers to a pair of inequalities connected either by the word “and” or the word “or.” Before we begin with the advanced work of solving these inequalities, let’s first spend a word or two (for purposes of review) discussing the solution of simple linear inequalities.

**Simple Linear Inequalities**

As in solving equations, you may add or subtract the same amount from both sides of an inequality.

**Property 1.** Let $a$ and $b$ be real numbers with $a < b$. If $c$ is any real number, then

$$a + c < b + c$$

and

$$a - c < b - c.$$  

This utility is equally valid if you replace the “less than” symbol with $>$, $\leq$, or $\geq$.

**Example 2.** Solve the inequality $x + 3 < 8$ for $x$.

Subtract 3 from both sides of the inequality and simplify.

$$x + 3 < 8$$

$$x + 3 - 3 < 8 - 3$$

$$x < 5$$

Thus, all real numbers less than 5 are solutions of the inequality. It is traditional to sketch the solution set of inequalities on a number line.

We can describe the solution set using set-builder and interval notation. The solution is

$$(-\infty, 5) = \{x : x < 5\}.$$  

An important concept is the idea of equivalent inequalities.

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**Equivalent Inequalities.** Two inequalities are said to be **equivalent** if and only if they have the same solution set.

Note that this definition is similar to the definition of equivalent equations. That is, two inequalities are equivalent if all of the solutions of the first inequality are also solutions of the second inequality, and vice-versa.

Thus, in Example 2, subtracting three from both sides of the original inequality produced an equivalent inequality. That is, the inequalities \( x + 3 < 8 \) and \( x < 5 \) have the same solution set, namely, all real numbers that are less than 5. It is no coincidence that the tools in Property 1 produce equivalent inequalities. Whenever you add or subtract the same amount from both sides of an inequality, the resulting inequality is equivalent to the original (they have the same solution set).

Let’s look at another example.

**Example 3.** Solve the inequality \( x - 5 \geq 4 \) for \( x \).

Add 5 to both sides of the inequality and simplify.

\[
\begin{align*}
x - 5 & \geq 4 \\
x - 5 + 5 & \geq 4 + 5 \\
x & \geq 9
\end{align*}
\]

Shade the solution on a number line.

In set-builder and interval notation, the solution is \( [9, \infty) = \{x : x \geq 9\} \)

You can also multiply or divide both sides by the same positive number.

**Property 4.** Let \( a \) and \( b \) be real numbers with \( a < b \). If \( c \) is a real positive number, then

\[
ac < bc
\]

and

\[
\frac{a}{c} < \frac{b}{c}
\]

Again, this utility is equally valid if you replace the “less than” symbol by \( >, \leq, \) or \( \geq \). The tools in Property 4 always produce equivalent inequalities.
Example 5. Solve the inequality $3x \leq -18$ for $x$.

Divide both sides of the inequality by 3 and simplify.

\[
\begin{align*}
3x & \leq -18 \\
\frac{3x}{3} & \leq \frac{-18}{3} \\
x & \leq -6
\end{align*}
\]

Sketch the solution on a number line.

\[
\begin{array}{c}
\text{solution}
\end{array}
\]

In set-builder and interval notation, the solution is

\[
(-\infty, -6] = \{x : x \leq -6\}.
\]

Thus far, there is seemingly no difference between the technique employed for solving inequalities and that used to solve equations. However, there is one important exception. Consider for a moment the true statement

\[
-2 < 6.
\]

If you multiply both sides of (6) by 3, you still have a true statement; i.e.,

\[
-6 < 18
\]

But if you multiply both sides of (6) by $-3$, you need to “reverse the inequality symbol” to maintain a true statement; i.e.,

\[
6 > -18.
\]

This discussion leads to the following property.

Property 7. Let $a$ and $b$ be real numbers with $a < b$. If $c$ is any real negative number, then

\[
ac > bc
\]

and

\[
\frac{a}{c} > \frac{b}{c}.
\]

Note that you “reverse the inequality symbol” when you multiply or divide both sides of an inequality by a negative number. Again, this utility is equally valid if you replace the “less than” symbol by $>$, $\leq$, or $\geq$. The tools in Property 7 always produce equivalent inequalities.
Example 8. Solve the inequality $-5x > 10$ for $x$.

Divide both sides of the inequality by $-5$ and reverse the inequality symbol. Simplify.

$$
\begin{align*}
-5x &> 10 \\
\frac{-5x}{-5} &< \frac{10}{-5} \\
x &< -2
\end{align*}
$$

Sketch the solution on a number line.

In set-builder and interval notation, the solution is

$$(-\infty, -2) = \{x : x < -2\}.$$

Compound Inequalities

We now turn our attention to the business of solving compound inequalities. In the previous section, we studied the subtleties of “and” and “or,” intersection and union, and looked at some simple compound inequalities. In this section, we build on those fundamentals and turn our attention to more intricate examples.

In this case, the best way of learning is by doing. Let’s start with an example.

Example 9. Solve the following compound inequality for $x$.

$$3 - 2x < -1 \quad \text{or} \quad 3 - 2x > 1 \quad (10)$$

First, solve each of the inequalities independently. With the first inequality, add $-3$ to both sides of the inequality, then divide by $-2$, reversing the inequality sign.

$$
\begin{align*}
3 - 2x &< -1 \\
-2x &< -4 \\
x &> 2
\end{align*}
$$

Shade the solution on a number line.

The exact same sequence of operations can be used to solve the second inequality.

$$
\begin{align*}
3 - 2x &> 1 \\
-2x &> -2 \\
x &< 1
\end{align*}
$$
Although you solve each side of the inequality independently, you will want to arrange your work as follows, stacking the number line solution for the first inequality above that of the second inequality.

\[
\begin{align*}
3 - 2x &< -1 \quad \text{or} \quad 3 - 2x > 1 \\
-2x &< -4 \quad \quad \\
\quad &x > 2 \\
-2x &> -2 \\
\quad &x < 1
\end{align*}
\]

Because the inequalities are connected with the word “or,” we need to take the union of these two number lines. That is, you want to shade every number from either set on a single number line, as shown in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{compound_inequality.png}
\caption{The solution of the compound inequality $3 - 2x < -1$ or $3 - 2x > 1$.}
\end{figure}

The solution, in interval and set-builder notation, is

\[
(-\infty, 1) \cup (2, \infty) = \{x : x < 1 \text{ or } x > 2\}.
\]

Let’s look at another example.

\textbf{Example 11.} Solve the following compound inequality for $x$.

\[
-1 < 3 - 2x < 1 \quad (12)
\]

Recall that $a < x < b$ is identical to the statement $x > a$ and $x < b$. Thus, we can write the compound inequality $-1 < 3 - 2x < 1$ in the form

\[
3 - 2x > -1 \quad \text{and} \quad 3 - 2x < 1. \quad (13)
\]

Solve each inequality independently, arranging your work as follows.

\[
\begin{align*}
3 - 2x &> -1 \quad \text{and} \quad 3 - 2x < 1 \\
-2x &> -4 \\
x &< 2 \\
\quad &x > 1
\end{align*}
\]

Shade the solution of each inequality on separate real lines, one atop the other.
Note the word “and” in our final statement $x < 2$ and $x > 1$. Thus, we must find the intersection of the two shaded solutions. These are the numbers that fall between 1 and 2, as shaded in Figure 2.

**Figure 2.** The solution of the compound inequality $-1 < 3 - 2x < 1$.

The solution, in both interval and set-builder notation, is

$$(1, 2) = \{x : 1 < x < 2\}.$$  

Note that we used the compact form of the compound inequality in our answer. We could just as well have used

$$(1, 2) = \{x : x > 1 \text{ and } x < 2\}.$$  

Both forms of set-builder notation are equally valid. You may use either one, but you must understand both.

**Alternative approach.** You might have noted that in solving the second inequality in (14), you found yourself repeating the identical operations used to solve the first inequality. That is, you subtracted 3 from both sides of the inequality, then divided both sides of the inequality by $-2$, reversing the inequality sign.

This repetition is annoying and suggests a possible shortcut in this particular situation. Instead of splitting the compound inequality (12) in two parts (as in (13)), let’s keep the inequality together, as in

$$-1 < 3 - 2x < 1. \quad (16)$$

Now, here are the rules for working with this form.
Property 17. When working with a compound inequality having the form

\[ a < x < b, \]  

you may add (or subtract) the same amount to (from) all three parts of the inequality, as in

\[ a + c < x + c < b + c \]  

or

\[ a - c < x - c < b - c. \]  

You may also multiply all three parts by the same positive number \( c > 0 \), as in

\[ ca < cx < cb. \]  

However, if you multiply all three parts by the same negative number \( c < 0 \), then don’t forget to reverse the inequality signs, as in

\[ ca > cx > cb. \]  

The rules for division are identical to the multiplication rules. If \( c > 0 \) (positive), then

\[ \frac{a}{c} < \frac{x}{c} < \frac{b}{c}. \]  

If \( c < 0 \) (negative), then reverse the inequality signs when you divide.

\[ \frac{a}{c} > \frac{x}{c} > \frac{b}{c}. \]  

Each of the tools in Property 17 always produce equivalent inequalities.

So, let’s return to the compound inequality (16) and subtract 3 from all three members of the inequality.

\[-1 < 3 - 2x < 1 \]
\[-1 - 3 < 3 - 2x - 3 < 1 - 3 \]
\[-4 < -2x < -2 \]

Next, divide all three members by \(-2\), reversing the inequality signs as you do so.

\[-4 < -2x < -2 \]
\[-4 > -2x > -2 \]
\[-2 > x > -2 \]
\[2 > x > 1 \]

It is conventional to change the order of this last inequality. By reading the inequality from right to left, we get
which describes the real numbers that are greater than 1 and less than 2. The solution is drawn on the following real line.

**Figure 3.** The solution of the compound inequality \(-1 < 3 - 2x < 1\).

Note that this is identical to the solution set on the real line in Figure 2. Note also that this second alternative method is more efficient, particularly if you do a bit of work in your head. Consider the following sequence where we subtract three from all three members, then divide all three members by \(-2\), reversing the inequality signs, then finally read the inequality in the opposite direction.

\[
\begin{align*}
-1 &< 3 - 2x < 1 \\
-4 &< -2x < -2 \\
2 &> x > 1 \\
1 &< x < 2
\end{align*}
\]

Let’s look at another example.

**Example 25.** Solve the following compound inequality for \(x\).

\[
-1 < x - \frac{x + 1}{2} \leq 2 \tag{26}
\]

First, let’s multiply all three members by 2, in order to clear the fractions.

\[
\begin{align*}
2(-1) &< 2 \left( x - \frac{x + 1}{2} \right) \leq 2(2) \\
-2 &< 2(x) - 2 \left( \frac{x + 1}{2} \right) \leq 4
\end{align*}
\]

Cancel. Note the use of parentheses, which is crucial when a minus sign is involved.

\[
\begin{align*}
-2 &< 2x - \frac{x + 1}{2} \leq 4 \\
-2 &< 2x - (x + 1) \leq 4
\end{align*}
\]

Distribute the minus sign and simplify.

\[
\begin{align*}
-2 &< 2x - x - 1 \leq 4 \\
-2 &< x - 1 \leq 4
\end{align*}
\]

Add 1 to all three members.

\[
-1 < x \leq 5
\]

This solution describes the real numbers that are greater than -1 and less than 5, including 5. That is, the real numbers that fall between -1 and 5, including 5, shaded on the real line in Figure 4.
The answer, described in both interval and set-builder notation, follows.

$$(-1, 5] = \{x : -1 < x \leq 5\}$$

Let’s look at another example.

**Example 27.** Solve the following compound inequality for $x$.

$$x \leq 2x - 3 \leq 5$$

Suppose that we try to isolate $x$ as we did in **Example 25**. Perhaps we would try adding $-x$ to all three members.

$$x \leq 2x - 3 \leq 5$$

$$x - x \leq 2x - 3 - x \leq 5 - x$$

$$0 \leq x - 3 \leq 5 - x$$

Well, that didn’t help much, just transferring the problem with $x$ to the other end of the inequality. Similar attempts will not help in isolating $x$. So, what do we do?

The solution is we split the inequality (with the word “and,” of course).

$$x \leq 2x - 3 \quad \text{and} \quad 2x - 3 \leq 5$$

We can solve the first inequality by subtracting $2x$ from both sides of the inequality, then multiplying both sides by $-1$, reversing the inequality in the process.

$$x \leq 2x - 3$$

$$-x \leq -3$$

$$x \geq 3$$

To solve the second inequality, add 3 to both sides, then divide both sides by 2.

$$2x - 3 \leq 5$$

$$2x \leq 8$$

$$x \leq 4$$

Of course, you’ll probably want to arrange your work as follows.

$$x \leq 2x - 3 \quad \text{and} \quad 2x - 3 \leq 5$$

$$-x \leq -3 \quad \quad 2x \leq 8$$

$$x \geq 3 \quad \quad x \leq 4$$

Thus, we need to shade on a number line all real numbers that are greater than or equal to 3 and less than or equal to 4, as shown in **Figure 5**.
Figure 5. When shading the solution of $x \leq 2x - 3 \leq 5$, we “fill-in” the end-points.

The solution, described in both interval and set-builder notation, is

$$[3, 4] = \{x : 3 \leq x \leq 4\}.$$