Prealgebra Textbook

Second Edition

Chapter 1

Department of Mathematics
College of the Redwoods

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Contents

Chapter 1

The Whole Numbers

Welcome to the study of prealgebra. In this first chapter of study, we will introduce the set of natural numbers, then follow with the set of whole numbers. We will then follow with a quick review of addition, subtraction, multiplication, and division skills involving whole numbers that are prerequisite for success in the study of prealgebra. Along the way we will introduce a number of properties of the whole numbers and show how that can be used to evaluate expressions involving whole number operations.

We will also define what is meant by *prime* and *composite* numbers, discuss a number of divisibility tests, then show how any composite number can be written uniquely as a product of prime numbers. This will lay the foundation for requisite skills with fractional numbers in later chapters.

Finally, we will introduce the concept of a *variable*, then introduce equations and technique required for their solution. We will use equations to model and solve a number of real-world applications along the way.

Let's begin the journey.

1.1 An Introduction to the Whole Numbers

A *set* is a collection of objects. If the set is finite, we can describe the set completely by simply listing all the objects in the set and enclosing the list in curly braces. For example, the set

$$S = \{\texttt{dog}, \texttt{cat}, \texttt{parakeet}\}$$

is the set whose members are "dog", "cat", and "parakeet." If the set is infinite, then we need to be more clever with our description. For example, the set of *natural numbers* (or *counting numbers*) is the set

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}.$$

Because this set is infinite (there are an infinite number of natural numbers), we can't list all of them. Instead, we list the first few then follow with "three dots," which essentially mean "etcetera." The implication is that the reader sees the intended pattern and can then intuit the remaining numbers in the set. Can you see that the next few numbers are 6, 7, 8, 9, etc.?

If we add the number zero to the set of natural numbers, then we have a set of numbers that are called the *whole numbers*.

The Whole Numbers. The set

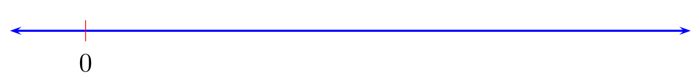
$$\mathbb{W} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

is called the set of *whole numbers*.

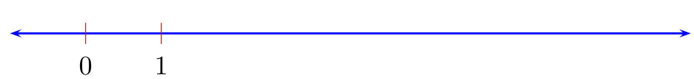
The whole numbers will be our focus in the remainder of this chapter.

Graphing numbers on the number line

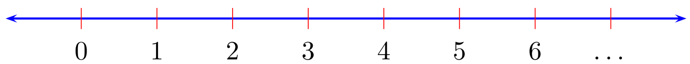
It is a simple matter to set up a correspondence between the whole numbers and points on a number line. First, draw a number line, then set a tick mark at zero.



The next step is to declare a unit length.



The remainder of the whole numbers now fall easily in place on the number line.



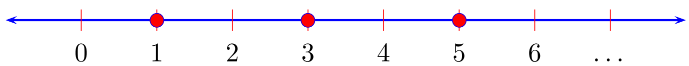
When asked to graph a whole number on a number line, shade in a solid dot at the position on the number line that corresponds to the given whole number.

You Try It!

EXAMPLE 1. Graph the whole numbers 1, 3, and 5 on the number line.

Graph the whole numbers 3, 4, and 6 on the number line.

Solution: Shade the numbers 1, 3, and 5 on the number line as solid dots.

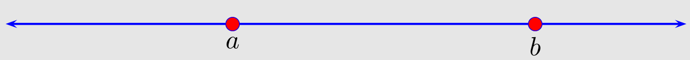


□

Ordering the whole numbers

Now that we have a correspondence between the whole numbers and points on the number line, we can *order* the whole numbers in a natural way. Note that as you move to the left along the number line, the numbers get smaller; as you move to the right, the numbers get bigger. This inspires the following definition.

Ordering the Whole Numbers. Suppose that a and b are whole numbers located on the number line so that the point representing the whole number a lies to the left of the point representing the whole number b.



Then the whole number a is "less than" the whole number b and write

$$a < b.$$

Alternatively, we can also say that the whole number b is"greater than" the whole number a and write

$$b > a.$$

Comparison Property: When comparing two whole numbers a and b, only one of three possibilities is true:

$$a < b \quad \text{or} \quad a = b \quad \text{or} \quad a > b.$$

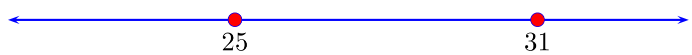
You Try It!

Compare the whole numbers 18 and 12.

Answer: $18 > 12$

EXAMPLE 2. Compare the whole numbers 25 and 31.

Solution: On the number line, 25 is located to the left of 31.



Therefore, 25 is less than 31 and write $25 < 31$. Alternatively, we could also note that 31 is located to the right of 25. Therefore, 31 is greater than 25 and write $31 > 25$. □

Expanded notation

The whole numbers

$$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

are called *digits* and are used to construct larger whole numbers. For example, consider the whole number 222 (pronounced "two hundred twenty two"). It is made up of three twos, but the position of each two describes a different meaning or value.

2	2	2
hundreds	tens	ones

- The first two is in the "hundreds" position and represents two hundreds or 200.
- The second two is in the "tens" position and represents two tens or 20.
- The third two is in the "ones" position and represents two ones or 2.

Consider the larger number 123,456,789. The following table shows the place value of each digit.

1	2	3	4	5	6	7	8	9
hundred millions	ten millions	millions	hundred thousands	ten thousands	thousands	hundreds	tens	ones
millions			thousands			ones		

In "expanded notation," we would write

$$\begin{aligned} &1 \text{ hundred million} + 2 \text{ ten millions} + 3 \text{ millions} + 4 \text{ hundred thousands} \\ &\quad + 5 \text{ ten thousands} + 6 \text{ thousands} + 7 \text{ hundreds} + 8 \text{ tens} + 9 \text{ ones.} \end{aligned}$$

We read the numeral 123,456,789 as "one hundred twenty three million, four hundred fifty six thousand, seven hundred eighty nine."

Let's look at another example.

You Try It!

EXAMPLE 3. Write the number 23,712 in expanded notation, then pronounce the result.

Solution: In expanded notation, 23,712 becomes

$$2 \text{ ten thousands} + 3 \text{ thousands} + 7 \text{ hundreds} + 1 \text{ ten} + 2 \text{ ones.}$$

This is pronounced "twenty three thousand, seven hundred twelve."

Write the number 54,615 in expanded notation. Pronounce the result.

□

You Try It!

EXAMPLE 4. Write the number 203,405 in expanded notation, then pronounce the result.

Solution: In expanded notation, 203,405 becomes

$$\begin{aligned} &2 \text{ hundred thousands} + 0 \text{ ten thousands} + 3 \text{ thousands} \\ &\quad + 4 \text{ hundreds} + 0 \text{ tens} + 5 \text{ ones.} \end{aligned}$$

Since 0 ten thousands is zero and 0 tens is also zero, this can also be written

$$2 \text{ hundred thousands} + 3 \text{ thousands} + 4 \text{ hundreds} + 5 \text{ ones.}$$

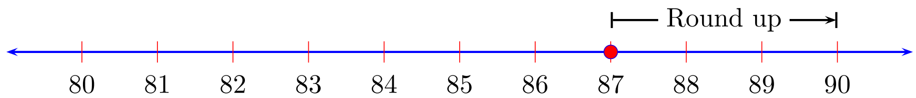
This is pronounced "two hundred three thousand, four hundred five."

Write the number $430,705$ in expanded notation. Pronounce the result.

□

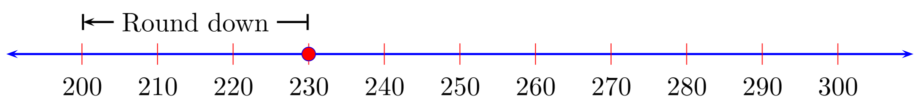
Rounding whole numbers

When less precision is needed, we round numbers to a particular place. For example, suppose a store owner needs approximately 87 boxes of ten-penny nails, but they can only be ordered in cartons containing ten boxes.



Note that 87 is located closer to 9 tens (or 90) than it is to 8 tens (or 80). Thus, rounded to the nearest ten, $87 \approx 90$ (87 approximately equals 90). The store owner decides that 90 boxes is probably a better fit for his needs.

On the other hand, the same store owner estimates that he will need 230 bags of peatmoss for his garden section.



Note that 230 is closer to 2 hundreds (or 200) than it is to 3 hundreds (or 300). The store owner worries that might have overestimated his need, so he rounds down to the nearest hundred, $230 \approx 200$ (230 approximately equals 200).

There is a simple set of rules to follow when rounding.

Rules for Rounding. To round a number to a particular place, follow these steps:

1. Mark the place you wish to round to. This is called the *rounding digit*.
2. Check the next digit to the right of your digit marked in step 1. This is called the *test digit*.
   a) If the test digit is greater than or equal to 5, add 1 to the rounding digit and replace all digits to the right of the rounding digit with zeros.
   b) If the test digit is less than 5, keep the rounding digit the same and replace all digits to the right of rounding digit with zeros.

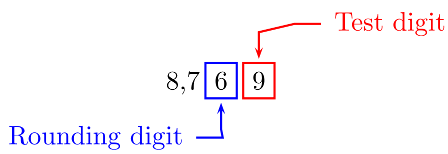
Let's try these rules with an example or two.

You Try It!

Round the number 9,443 to the nearest ten.

EXAMPLE 5. Round the number 8,769 to the nearest ten.

Solution: Mark the rounding and test digits.



The test digit is greater than 5. The "Rules for Rounding" require that we add 1 to the rounding digit, then make all digits to the right of the rounding digit zeros. Thus, rounded to the nearest ten,

$$8,769 \approx 8,770.$$

That is, 8,769 is *approximately equal* to 8,770.

Answer: 9,440

Mathematical Notation. The symbol

$$\approx$$

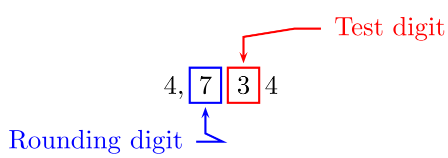
means *approximately equal*.

You Try It!

EXAMPLE 6. Round the number 4,734 to the nearest hundred.

Round the number 6,656 to the nearest hundred.

Solution: Mark the rounding and test digits.



The test digit is less than 5. The "Rules for Rounding" require that we keep the rounding digit the same, then make all digits to the right of the rounding digit zeros. Thus, rounded to the nearest hundred,

$$4,734 \approx 4,700.$$

Answer: $6,700$

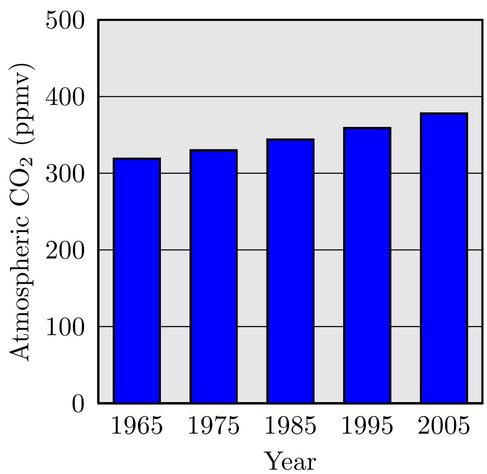
Year	1965	1975	1985	1995	2005
Atmospheric CO_2	319	330	344	359	378

Table 1.1: Atmospheric CO_2 values (ppmv) derived from in situ air samples collected at Mauna Loa, Hawaii, USA.

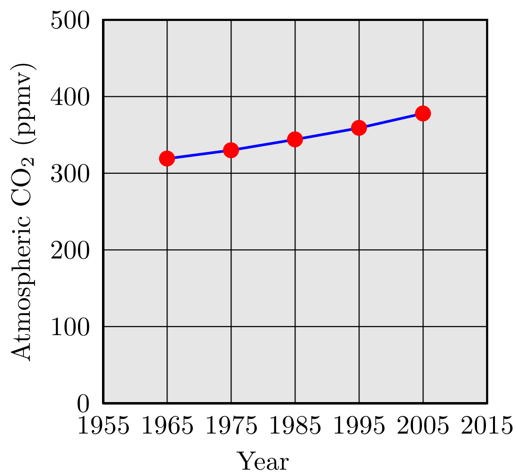
Tables and graphs

Reading data in graphical form is an important skill. The data in **Table 1.1** provides measures of the carbon dioxide content (CO_2) in the atmosphere, gathered in the month of January at the observatory atop Mauna Loa in Hawaii.

In **Figure 1.1(a)**, a *bar graph* is used to display the carbon dioxide measurements. The year the measurement was taken is placed on the horizontal axis, and the height of each bar equals the amount of carbon dioxide in the atmosphere during that year.



(a) Bar graph.



(b) Line graph.

Figure 1.1: Using graphs to examine carbon dioxide data.

In **Figure 1.1(b)**, a *line graph* is used to display the carbon dioxide measurements. Again, the dates of measurement are placed on the horizontal axis, and the amount of carbon dioxide in the atmosphere is placed on the vertical axis. Instead of using the height of a bar to represent the carbon dioxide measurement, we place a dot at a height that represents the carbon monoxide content. Once each data point is plotted, we connect consecutive data points with line segments.

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-12, sketch the given whole numbers on a number line, then arrange them in order, from smallest to largest.

1. 2, 8, and 4

2. 2, 7, and 4

3. 1, 8, and 2

4. 0, 4, and 3

5. 0, 4, and 1

6. 3, 6, and 5

7. 4, 9, and 6

8. 2, 4, and 3

9. 0, 7, and 4

10. 2, 8, and 6

11. 1, 6, and 5

12. 0, 9, and 5

In Exercises 13-24, create a number line diagram to determine which of the two given statements is true.

13. $3 < 8$ or $3 > 8$

14. $44 < 80$ or $44 > 80$

15. $59 < 24$ or $59 > 24$

16. $15 < 11$ or $15 > 11$

17. $0 < 74$ or $0 > 74$

18. $11 < 18$ or $11 > 18$

19. $1 < 81$ or $1 > 81$

20. $65 < 83$ or $65 > 83$

21. $43 < 1$ or $43 > 1$

22. $62 < 2$ or $62 > 2$

23. $43 < 28$ or $43 > 28$

24. $73 < 21$ or $73 > 21$

---

25. Which digit is in the thousands column of the number 2,054,867,372?

26. Which digit is in the hundreds column of the number 2,318,999,087?

27. Which digit is in the hundred thousands column of the number 8,311,900,272?

28. Which digit is in the tens column of the number 1,143,676,212?

29. Which digit is in the hundred millions column of the number 9,482,616,000?

30. Which digit is in the hundreds column of the number 375,518,067?

31. Which digit is in the ten millions column of the number 5,840,596,473?

32. Which digit is in the hundred thousands column of the number 6,125,412,255?

33. Which digit is in the hundred millions column of the number 5,577,422,501?

34. Which digit is in the thousands column of the number 8,884,966,835?

35. Which digit is in the tens column of the number 2,461,717,362?

36. Which digit is in the ten millions column of the number 9,672,482,548?

37. Round the number $93,857$ to the nearest thousand.

38. Round the number $56,872$ to the nearest thousand.

39. Round the number $9,725$ to the nearest ten.

40. Round the number $6,815$ to the nearest ten.

41. Round the number $58,739$ to the nearest hundred.

42. Round the number $93,146$ to the nearest hundred.

43. Round the number $2,358$ to the nearest ten.

44. Round the number $8,957$ to the nearest ten.

45. Round the number $39,756$ to the nearest thousand.

46. Round the number $24,965$ to the nearest thousand.

47. Round the number $5,894$ to the nearest ten.

48. Round the number $3,281$ to the nearest ten.

49. Round the number $56,123$ to the nearest hundred.

50. Round the number $49,635$ to the nearest hundred.

51. Round the number $5,483$ to the nearest ten.

52. Round the number $9,862$ to the nearest ten.

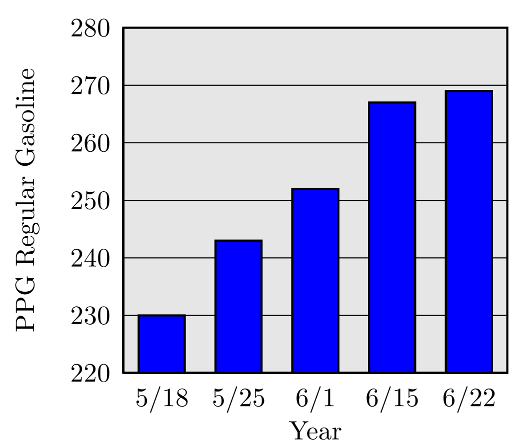
53. According to the U.S. Census Bureau, the estimated population of the US is 304,059,724 as of July 2008. Round to the nearest hundred thousand.

54. According to the U.S. Census Bureau, the estimated population of California is 36,756,666 as of July 2008. Round to the nearest hundred thousand.

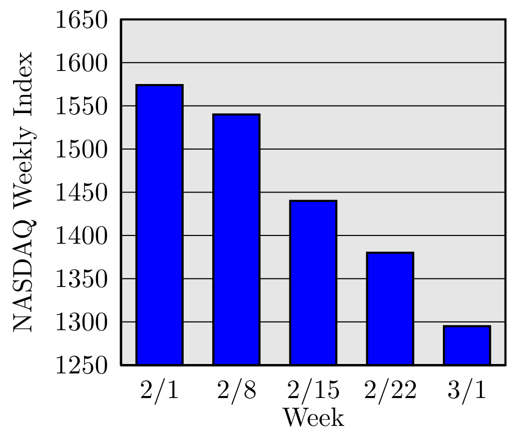
55. According to the U.S. Census Bureau, the estimated population of Humboldt County is 129,000 as of July 2008. Round to the nearest ten thousand.

56. According to the U.S. Census Bureau, the estimated population of the state of Alasks was 686,293 as of July 2008. Round to the nearest ten thousand.

57. The following bar chart shows the average price (in cents) of one gallon of regular gasoline in the United States over five consecutive weeks in 2009, running from May 18 (5/18) through June 22 (6/22). What was the price (in cents) of one gallon of regular gasoline on June 1, 2009?



58. The following bar chart shows the average weekly NASDAQ index for five consecutive weeks in 2009, beginning with week starting February 1 (2/1) and ending with the week starting March 1 (3/1). What was the average NASDAQ index for the week starting February 8, 2009?



59. The population of Humboldt County is broken into age brackets in the following table. *Source: WolframAlpha.*

Age in years	Number
under 5	7,322
5-18	26,672
18-65	78,142
over 65	16,194

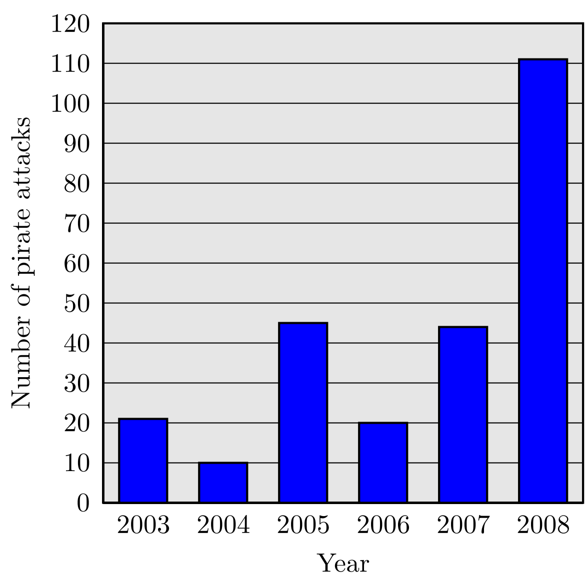
Create a bar chart for this data set with one bar for each age category.

60. The five cities with the largest number of reported violent crimes in the year 2007 are reported in the following table. *Source: Wikipedia.*

City	Violent Crimes
Detroit	2,289
St. Louis	2,196
Memphis	1,951
Oakland	1,918
Baltimore	1,631

Create a bar chart for this data set with one bar for each city.

61. The following bar chart tracks pirate attacks off the coast of Somalia.

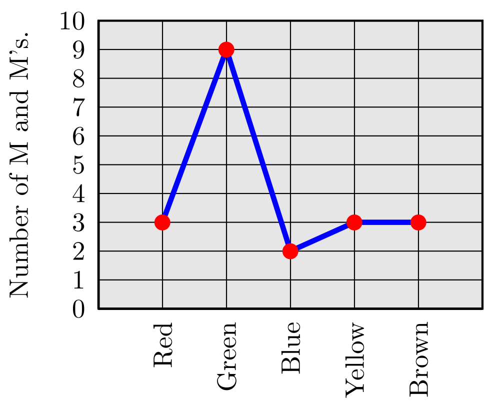


Source: ICC International Maritime Bureau, AP Times-Standard, 4/15/2009

a) How many pirate attacks were there in 2003?

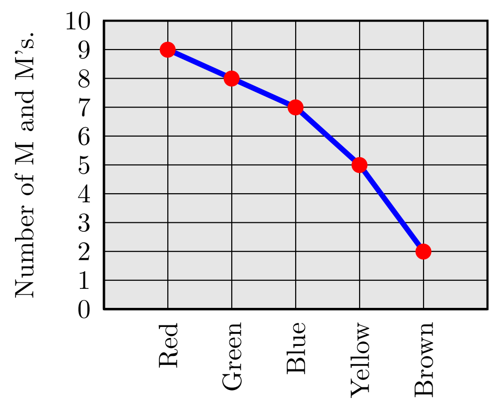
b) How many pirate attacks were there in 2008?

62. A team of students separated a small bowl of M and M's into five piles by color. The following line plot indicates the number of M and M's of each color.



How many red M and M's were in the bowl?

63. A team of students separated a small bowl of M and M's into five piles by color. The following line plot indicates the number of M and M's of each color.



How many red M and M's were in the bowl?

64. A team of students separated a small bowl of M and M's into five piles by color. The following table indicates the number of M and M's of each color.

Color	Number
Red	5
Green	9
Blue	7
Yellow	2
Brown	3

Create a lineplot for the M and M data. On the horizontal axis, arrange the colors in the same order as presented in the table above.

65. A team of students separated a small bowl of M and M's into five piles by color. The following table indicates the number of M and M's of each color.

Color	Number
Red	3
Green	7
Blue	2
Yellow	4
Brown	9

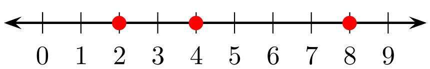
Create a lineplot for the M and M data. On the horizontal axis, arrange the colors in the same order as presented in the table above.

66. Salmon count. The table shows the number of adult coho salmon returning to the Shasta River over the past four years. Round the salmon count for each year to the nearest ten. *Times-Standard Shasta River coho rescue underway.*

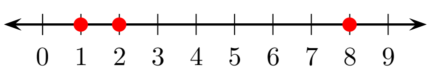
Year	Salmon count
2007	300
2008	31
2009	9
2010	4

❧ ❧ ❧ Answers ☙ ☙ ☙

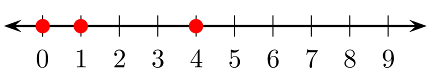
1. Smallest to largest: 2, 4, and 8.



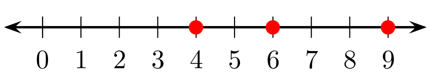
3. Smallest to largest: 1, 2, and 8.



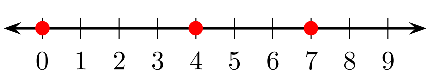
5. Smallest to largest: 0, 1, and 4.



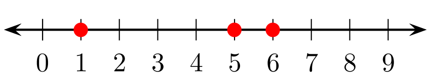
7. Smallest to largest: 4, 6, and 9.



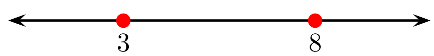
9. Smallest to largest: 0, 4, and 7.



11. Smallest to largest: 1, 5, and 6.

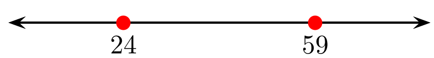


13.



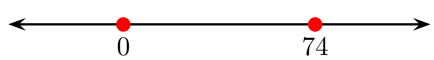
Therefore, $3 < 8$.

15.



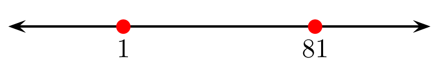
Therefore, $59 > 24$.

17.



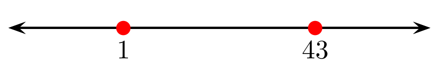
Therefore, $0 < 74$.

19.



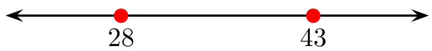
Therefore, $1 < 81$.

21.



Therefore, $43 > 1$.

23.



Therefore, $43 > 28$.

25. 7

27. 9

29. 4

31. 4

33. 5

35. 6

37. 94000

39. 9730

41. 58700

43. 2360

45. 40000

47. 5890

49. 56100

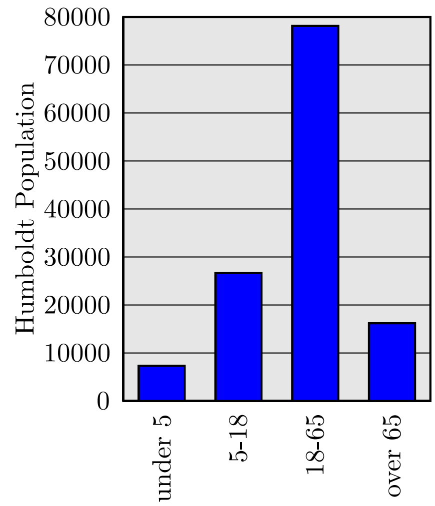
51. 5480

53. 304,100,000

55. 130,000

57. Approximately 252 cents

59.

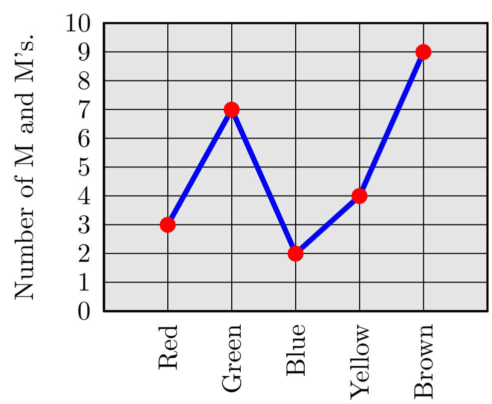


61. a) Approximately 21

b) Approximately 111

63. 9

65.



1.2 Adding and Subtracting Whole Numbers

In the expression $3 + 4$, which shows the sum of two whole numbers, the whole numbers 3 and 4 are called *addends* or *terms*. We can use a visual approach to find the sum of 3 and 4. First, construct a number line as shown in Figure 1.2.

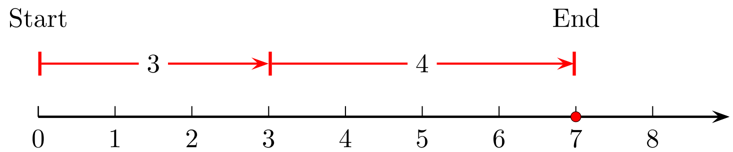


Figure 1.2: Adding whole numbers on a number line.

To add 3 and 4, proceed as follows.

1. Start at the number 0, then draw an arrow 3 units to the right, as shown in Figure 1.2. This arrow has magnitude (length) three and represents the whole number 3.
2. Draw a second arrow of length four, starting at the end of the first arrow representing the number 3. This arrow has magnitude (length) four and represents the whole number 4.
3. The sum of 3 and 4 could be represented by an arrow that starts at the number 0 and ends at the number 7. However, we prefer to mark this sum on the number line as a solid dot at the whole number 7. This number represents the sum of the whole numbers 3 and 4.

The Commutative Property of Addition

Let's change the order in which we add the whole numbers 3 and 4. That is, let's find the sum $4 + 3$ instead.

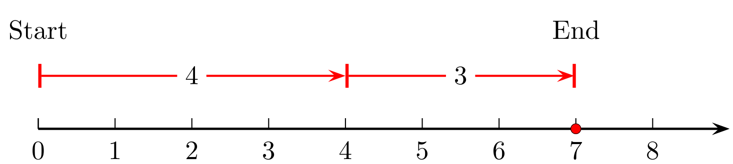


Figure 1.3: Addition is commutative; i.e., order doesn't matter.

As you can see in Figure 1.3, we start at zero then draw an arrow of length four, followed by an arrow of length three. However, the result is the same; i.e., $4 + 3 = 7$.

Thus, the order in which we add three and four does not matter; that is,

$$3 + 4 = 4 + 3.$$

This property of addition of whole numbers is known as the *commutative property* of addition.

The Commutative Property of Addition. Let a and b represent two whole numbers. Then,

$$a + b = b + a.$$

Grouping Symbols

In mathematics, we use *grouping symbols* to affect the order in which an expression is evaluated. Whether we use parentheses, brackets, or curly braces, the expression inside any pair of grouping symbols must be evaluated first. For example, note how we first evaluate the sum in the parentheses in the following calculation.

$$\begin{aligned} (3+4)+5 &= 7+5 \\ &= 12 \end{aligned}$$

The rule is simple: Whatever is inside the parentheses is evaluated first.

Writing Mathematics. When writing mathematical statements, follow the mantra:

One equal sign per line.

We can use brackets instead of parentheses.

$$\begin{aligned} 5+[7+9] &= 5+16 \\ &= 21 \end{aligned}$$

Again, note how the expression inside the brackets is evaluated first.

We can also use curly braces instead of parentheses or brackets.

$$\begin{aligned} \{2+3\}+4 &= 5+4 \\ &= 9 \end{aligned}$$

Again, note how the expression inside the curly braces is evaluated first.

If grouping symbols are *nested*, we evaluate the innermost parentheses first. For example,

$$\begin{aligned} 2+[3+(4+5)] &= 2+[3+9] \\ &= 2+12 \\ &= 14. \end{aligned}$$

Grouping Symbols. Use parentheses, brackets, or curly braces to delimit the part of an expression you want evaluated first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.

The Associative Property of Addition

Consider the evaluation of the expression (2+3)+4. We evaluate the expression in parentheses first.

$$\begin{aligned}(2+3)+4 &= 5+4\\ &= 9\end{aligned}$$

Now, suppose we change the order of addition to $2+(3+4)$. Then,

$$\begin{aligned}2+(3+4) &= 2+7\\ &= 9.\end{aligned}$$

Although the grouping has changed, the result is the same. That is,

$$(2+3)+4 = 2+(3+4).$$

This property of addition of whole numbers is called the *associate property* of addition.

Associate Property of Addition. Let a, b, and c represent whole numbers. Then,

$$(a+b)+c = a+(b+c).$$

Because of the associate property of addition, when presented with a sum of three numbers, whether you start by adding the first two numbers or the last two numbers, the resulting sum is the same.

The Additive Identity

Imagine a number line visualization of the sum of four and zero; i.e., $4+0$.

In Figure 1.4, we start at zero, then draw an arrow of magnitude (length) four pointing to the right. Now, at the end of this arrow, attach a second arrow of length zero. Of course, that means that we remain right where we are, at 4. Hence the shaded dot at 4 is the sum. That is, $4+0=4$.

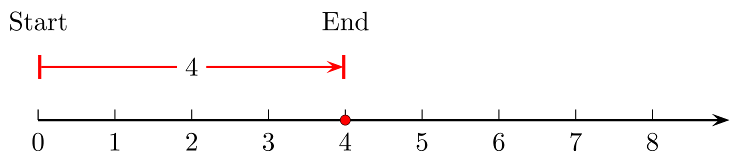


Figure 1.4: Adding zero to four.

The Additive Identity Property. The whole number zero is called the *additive identity.* If a is any whole number, then

$$a + 0 = a.$$

The number zero is called the additive identity because if you add zero to any number, you get the identical number back.

Adding Larger Whole Numbers

For completeness, we include two examples of adding larger whole numbers. Hopefully, the algorithm is familiar from previous coursework.

You Try It!

Simplify: $1,286 + 349$

EXAMPLE 1. Simplify: $1,234 + 498$.

Solution. Align the numbers vertically, then add, starting at the furthest column to the right. Add the digits in the ones column, $4 + 8 = 12$. Write the 2, then carry a 1 to the tens column. Next, add the digits in the tens column, $3 + 9 = 12$, add the carry to get 13, then write the 3 and carry a 1 to the hundreds column. Continue in this manner, working from right to left.

$$\begin{array}{ccccc} & & 1 & 1 & \\ & 1 & 2 & 3 & 4 \\ + & & 4 & 9 & 8 \\ \hline & 1 & 7 & 3 & 2 \end{array}$$

Therefore, $1,234 + 498 = 1,732$.

Answer: 1,635

Add three or more numbers in the same manner.

You Try It!

Simplify: $256 + 342 + 283$

EXAMPLE 2. Simplify: $256 + 322 + 418$.

Solution. Align the numbers vertically, then add, starting at the furthest column to the right. Add the digits in the ones column, $6+2+8=16$. Write the 6, then carry a 1 to the tens column. Continue in this manner, working from right to left.

$$\begin{array}{r} {\scriptstyle 1} \\ 2\ 5\ 6 \\ 3\ 2\ 2 \\ +\ 4\ 1\ 8 \\ \hline 9\ 9\ 6 \end{array}$$

Therefore, $256+322+418=996$.

Answer: 881

Subtraction of Whole Numbers

The key idea is this: *Subtraction is the opposite of addition.* For example, consider the difference $7-4$ depicted on the number line in Figure 1.5.

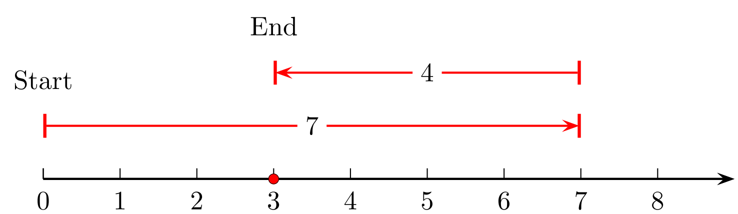


Figure 1.5: Subtraction means *add the opposite*.

If we were adding 7 and 4, we first draw an arrow starting at zero pointing to the right with magnitude (length) seven. Then, to add 4, we would draw a second arrow of magnitude (length) 4, attached to the end of the first arrow and pointing to the right.

However, because subtraction is the *opposite* of addition, in Figure 1.5 we attach an arrow of magnitude (length) four to the end of the first arrow, but *pointing in the opposite direction* (to the left). Note that this last arrow ends at the answer, which is a shaded dot on the number line at 3. That is, $7-4=3$.

Note that subtraction is **not commutative**; that is, it make no sense to say that $7-5$ is the same as $5-7$.

Subtraction is **not associative**. It is not the case that $(9-5)-2$ is the same as $9-(5-2)$. On the one hand,

$$\begin{aligned}(9-5)-2 &= 4-2 \\ &= 2,\end{aligned}$$

but

$$\begin{aligned} 9 - (5 - 2) &= 9 - 3 \\ &= 6. \end{aligned}$$

Subtracting Larger Whole Numbers

Much as we did with adding larger whole numbers, to subtract two large whole numbers, align them vertically then subtract, working from right to left. You may have to "borrow" to complete the subtraction at any step.

You Try It!

Simplify: $5,635 - 288$.

EXAMPLE 3. Simplify: $1,755 - 328$.

Solution. Align the numbers vertically, then subtract, starting at the ones column, then working right to left. At the ones column, we cannot subtract 8 from 5, so we borrow from the previous column. Now, 8 from 15 is 7. Continue in this manner, working from right to left.

$$\begin{array}{rcccc} & 1 & 7 & \overset{4}{\not 5} & {}^{1}5 \\ - & & 3 & 2 & 8 \\ \hline & 1 & 4 & 2 & 7 \end{array}$$

Answer: 5,347

Therefore, $1,755 - 328 = 1,427$.

□

Order of Operations

In the absence of grouping symbols, it is important to understand that addition holds no precedence over subtraction, and vice-versa.

> Perform all additions and subtractions in the order presented, moving left to right.

Let's look at an example.

You Try It!

Simplify: $25 - 10 + 8$.

EXAMPLE 4. Simplify the expression $15 - 8 + 4$.

Solution. This example can be trickier than it seems. However, if we follow the rule (perform all additions and subtractions in the order presented, moving left to right), we should have no trouble. First comes fifteen minus eight, which is seven. Then seven plus four is eleven.

$$\begin{aligned} 15 - 8 + 4 &= 7 + 4 \\ &= 11. \end{aligned}$$

Answer: 23

Caution! Incorrect answer ahead! Note that it is possible to arrive at a different (but incorrect) answer if we favor addition over subtraction in Example 4. If we first add eight and four, then $15 - 8 + 4$ becomes $15 - 12$, which is 3. However, note that **this is incorrect**, because it violates the rule "perform all additions and subtractions in the order presented, moving left to right."

Applications — Geometry

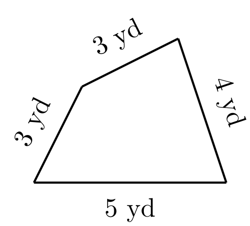
There are any number of applications that require a sum or difference of whole numbers. Let's examine a few from the world of geometry.

Perimeter of a Polygon. In geometry a polygon is a plane figure made up of a closed path of a finite sequence of segments. The segments are called the *edges* or *sides* of the polygon and the points where two edges meet are called the *vertices* of the polygon. The *perimeter* of any polygon is the sum of the lengths of its sides.

You Try It!

EXAMPLE 5. A quadrilateral is a polygon with four sides. Find the perimeter of the quadrilateral shown below, where the sides are measured in yards.

A quadrilateral has sides that measure 4 in., 3 in., 5 in., and 5 in. Find the perimeter.



Solution. To find the perimeter of the quadrilateral, find the sum of the lengths of the sides.

$$\text{Perimeter} = 3 + 3 + 4 + 5 = 15$$

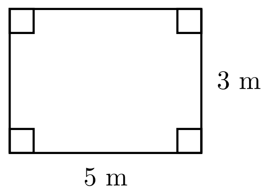
Hence, the perimeter of the quadrilateral is 15 yards.

Answer: 17 inches

You Try It!

A rectangle has length 12 meters and width 8 meters. Find its perimeter.

EXAMPLE 6. A quadrilateral (four sides) is a *rectangle* if all four of its angles are right angles. It can be shown that the opposite sides of a rectangle must be equal. Find the perimeter of the rectangle shown below, where the sides of the rectangle are measured in meters.



Solution. To find the perimeter of the rectangle, find the sum of the four sides. Because opposite sides have the same length, we have two sides of length 5 meters and two sides of length 3 meters. Hence,

$$\text{Perimeter} = 5 + 3 + 5 + 3 = 16.$$

Thus, the perimeter of the rectangle is 16 meters.

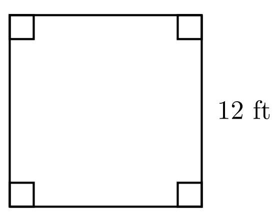
Answer: 40 meters

□

You Try It!

A square has a side that measures 18 centimeters. Find its perimeter.

EXAMPLE 7. A quadrilateral (four sides) is a *square* if all four of its sides are equal and all four of its angles are right angles. Pictured below is a square having a side of length 12 feet. Find the perimeter of the square.



Solution. Because the quadrilateral is a square, all four sides have the same length, namely 12 feet. To find the perimeter of the square, find the sum of the four sides.

$$\text{Perimeter} = 12 + 12 + 12 + 12 = 48$$

Hence, the perimeter of the square is 48 feet.

Answer: 72 centimeters

□

Application — Alternative Fuels

Automobiles that run on alternative fuels (other than gasoline) have increased in the United States over the years.

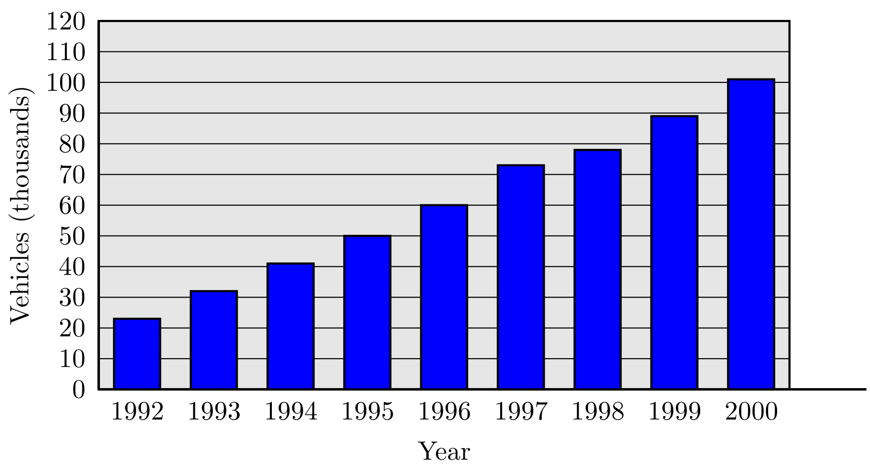
You Try It!

EXAMPLE 8. Table 1.2 show the number of cars (in thousands) running on compressed natural gas versus the year. Create a bar chart showing the number of cars running on compressed natural gas versus the year.

Year	1992	1993	1994	1995	1996	1997	1998	1999	2000
Number	23	32	41	50	60	73	78	89	101

Table 1.2: Number of vehicles (in thousands) running on compressed natural gas.

Solution. Place the years on the horizontal axis. At each year, sketch a bar having height equal to the number of cars in that year that are running on compressed natural gas. Scale the vertical axis in thousands.



□

The following table shows the number of hybrid cars (in thousands) by country.

Country	Number
U.S.	279
Japan	77
Canada	17
U.K.	14
Netherlands	11

Create a bar chart showing the number of cars versus the country of use.

You Try It!

The following table show Alphonso's percentage scores on his examinations in mathematics.

Exam	Percentage
Exam #1	52
Exam #2	45
Exam #3	72
Exam #4	889
Exam #5	76

Construct a line graph of Alphonso's exam scores versus exam number.

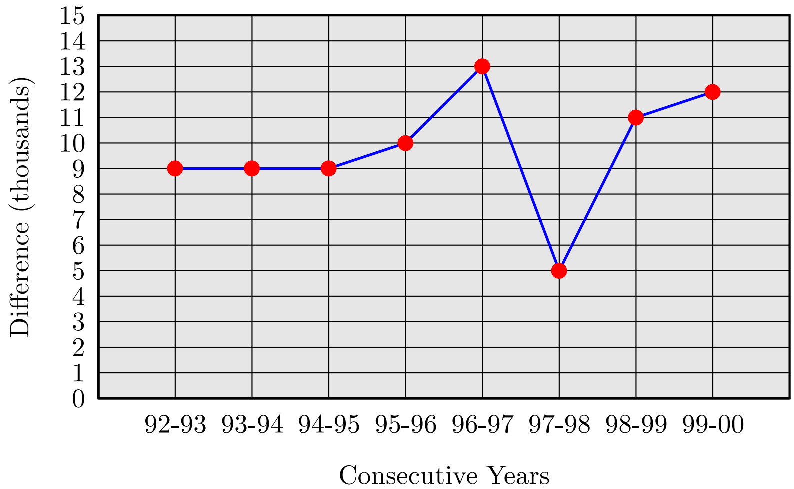
EXAMPLE 9. Using the data in Table 1.2, create a table that shows the differences in consecutive years, then create a line plot of the result. In what consecutive years did the United States see the greatest increase in cars powered by compressed natural gas?

Solution. Table 1.3 shows the differences in consecutive years.

Years	92-93	93-94	94-95	95-96	96-97	97-98	98-99	99-00
Difference	9	9	9	10	13	5	11	12

Table 1.3: Showing the differences in vehicles in consecutive years.

Next, craft a line graph. Place consecutive years on the horizontal axis. At each consecutive year pair, plot a point at a height equal to the difference in alternative fuel vehicles. Connect the points with straight line segments.



Note how the line graph makes it completely clear that the greatest increase in vehicles powered by compressed natural gas occurred in the consecutive years 1996-1997, an increase of 13,000 vehicles.

□

❧ ❧ ❧ Exercises ☙ ☙ ☙

1. Sketch a number line diagram depicting the sum $3 + 2$, as shown in Figure 1.2 in the narrative of this section.

2. Sketch a number line diagram depicting the sum $3 + 5$, as shown in Figure 1.2 in the narrative of this section.

3. Sketch a number line diagram depicting the sum $3 + 4$, as shown in Figure 1.2 in the narrative of this section.

4. Sketch a number line diagram depicting the sum $2 + 4$, as shown in Figure 1.2 in the narrative of this section.

5. Sketch a number line diagram depicting the sum $4 + 2$, as shown in Figure 1.2 in the narrative of this section.

6. Sketch a number line diagram depicting the sum $4 + 3$, as shown in Figure 1.2 in the narrative of this section.

7. Sketch a number line diagram depicting the sum $2 + 5$, as shown in Figure 1.2 in the narrative of this section.

8. Sketch a number line diagram depicting the sum $4 + 5$, as shown in Figure 1.2 in the narrative of this section.

9. Sketch a number line diagram depicting the sum $4 + 4$, as shown in Figure 1.2 in the narrative of this section.

10. Sketch a number line diagram depicting the sum $3 + 3$, as shown in Figure 1.2 in the narrative of this section.

In Exercises 11-28, determine which property of addition is depicted by the given identity.

11. $28 + 0 = 28$

12. $53 + 0 = 53$

13. $24 + 0 = 24$

14. $93 + 0 = 93$

15. $(51 + 66) + 88 = 51 + (66 + 88)$

16. $(90 + 96) + 4 = 90 + (96 + 4)$

17. $64 + 39 = 39 + 64$

18. $68 + 73 = 73 + 68$

19. $(70 + 27) + 52 = 70 + (27 + 52)$

20. $(8 + 53) + 81 = 8 + (53 + 81)$

21. $79 + 0 = 79$

22. $42 + 0 = 42$

23. $10 + 94 = 94 + 10$

24. $55 + 86 = 86 + 55$

25. $47 + 26 = 26 + 47$

26. $62 + 26 = 26 + 62$

27. $(61 + 53) + 29 = 61 + (53 + 29)$

28. $(29 + 96) + 61 = 29 + (96 + 61)$

---

29. Sketch a number line diagram depicting the difference $8-2$, as shown in Figure 1.5 in the narrative of this section.

30. Sketch a number line diagram depicting the difference $8-4$, as shown in Figure 1.5 in the narrative of this section.

31. Sketch a number line diagram depicting the difference $7-2$, as shown in Figure 1.5 in the narrative of this section.

32. Sketch a number line diagram depicting the difference $9-5$, as shown in Figure 1.5 in the narrative of this section.

33. Sketch a number line diagram depicting the difference $7-4$, as shown in Figure 1.5 in the narrative of this section.

34. Sketch a number line diagram depicting the difference $6-4$, as shown in Figure 1.5 in the narrative of this section.

35. Sketch a number line diagram depicting the difference $9-4$, as shown in Figure 1.5 in the narrative of this section.

36. Sketch a number line diagram depicting the difference $6-5$, as shown in Figure 1.5 in the narrative of this section.

37. Sketch a number line diagram depicting the difference $8-5$, as shown in Figure 1.5 in the narrative of this section.

38. Sketch a number line diagram depicting the difference $9-3$, as shown in Figure 1.5 in the narrative of this section.

In Exercises 39-50, simplify the given expression.

39. $16 - 8 + 2$

40. $17 - 3 + 5$

41. $20 - 5 + 14$

42. $14 - 5 + 6$

43. $15 - 2 + 5$

44. $13 - 4 + 2$

45. $12 - 5 + 4$

46. $19 - 4 + 13$

47. $12 - 6 + 4$

48. $13 - 4 + 18$

49. $15 - 5 + 8$

50. $13 - 3 + 11$

In Exercises 51-58, the width W and length L of a rectangle are given. Find the perimeter P of the rectangle.

51. $W = 7$ in, $L = 9$ in

52. $W = 4$ in, $L = 6$ in

53. $W = 8$ in, $L = 9$ in

54. $W = 5$ in, $L = 9$ in

55. $W = 4$ cm, $L = 6$ cm

56. $W = 5$ in, $L = 8$ in

57. $W = 4$ cm, $L = 7$ cm

58. $W = 4$ in, $L = 9$ in

In Exercises 59-66, the length s of a side of a square is given. Find the perimeter P of the square.

59. $s = 25$ cm

60. $s = 21$ in

61. $s = 16$ cm

62. $s = 10$ in

63. $s = 18$ in

64. $s = 7$ in

65. $s = 3$ in

66. $s = 20$ in

In Exercises 67-86, find the sum.

67. $3005 + 5217$

68. $1870 + 5021$

69. $575 + 354 + 759$

70. $140 + 962 + 817$

71. $472 + (520 + 575)$

72. $318 + (397 + 437)$

73. $274 + (764 + 690)$

74. $638 + (310 + 447)$

75. $8583 + 592$

76. $5357 + 9936$

77. $899 + 528 + 116$

78. $841 + 368 + 919$

79. $(466 + 744) + 517$

80. $(899 + 996) + 295$

81. $563 + 298 + 611 + 828$

82. $789 + 328 + 887 + 729$

83. $607 + 29 + 270 + 245$

84. $738 + 471 + 876 + 469$

85. $(86 + 557) + 80$

86. $(435 + 124) + 132$

In Exercises 87-104, find the difference.

87. $3493 - 2034 - 227$

88. $3950 - 1530 - 2363$

89. $8338 - 7366$

90. $2157 - 1224$

91. $2974 - 2374$

92. $881 - 606$

93. $3838 - (777 - 241)$

94. $8695 - (6290 - 4233)$

95. $5846 - 541 - 4577$

96. $5738 - 280 - 4280$

97. $3084 - (2882 - 614)$

98. $1841 - (217 - 28)$

99. $2103 - (1265 - 251)$

100. $1471 - (640 - 50)$

101. $9764 - 4837 - 150$

102. $9626 - 8363 - 1052$

103. $7095 - 226$

104. $4826 - 1199$

105. **Water Subsidies.** Since the drought began in 2007, California farms have received \$79 million in water subsidies. California cotton and rice farmers received an additional \$439 million. How much total water subsidies have farmers received? *Associated Press Times-Standard 4/15/09*

106. **War Budget.** The 2010 Federal budget allocates \$534 billion for the Department of Defense base programs and an additional \$130 billion for the nation's two wars. How much will the Department of Defense receive altogether? *Associated Press Times-Standard 5/8/09*

107. **Sun Frost.** Arcata, CA is home to Sun Frost, a manufacturer of highly efficient refrigerators and freezers. The AC model RF12 refrigerator/freezer costs \$2,279 while an R16 model refrigerator/freezer costs \$3,017. How much more does the R16 model cost? *Source: www.sunfrost.com/retail_pricelist.html*

108. **Shuttle Orbit.** The space shuttle usually orbits at 250 miles above the surface of the earth. To service the Hubble Space Telescope, the shuttle had to go to 350 miles above the surface. How much higher did the shuttle have to orbit?

109. **Earth's Orbit.** Earth orbits the sun in an ellipse. When earth is at its closest to the sun, called *perihelion*, earth is about 147 million kilometers. When earth is at its furthest point from the sun, called *aphelion*, earth is about 152 million kilometers from the sun. What's the difference in millions of kilometers between aphelion and perihelion?

110. **Pluto's Orbit.** Pluto's orbit is highly eccentric. Find the difference between Pluto's closest approach to the sun and Pluto's furthest distance from the sun if Pluto's perihelion (closest point on its orbit about the sun) is about 7 billion kilometers and its aphelion (furthest point on its orbit about the sun) is about 30 billion kilometers.

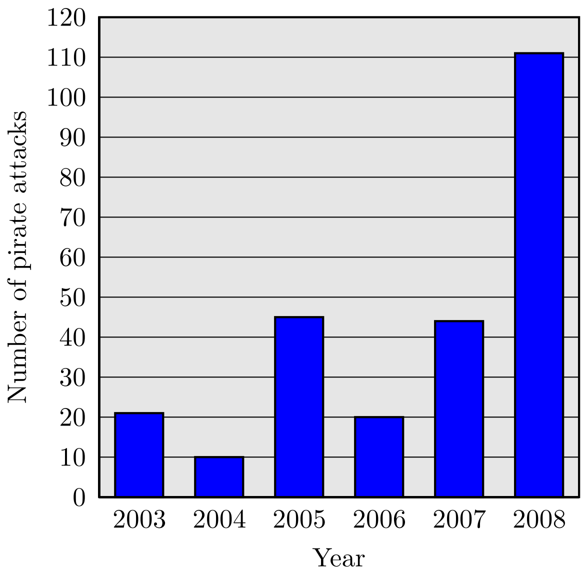
111. **Sunspot Temperature.** The surface of the sun is about 10,000 degrees Fahrenheit. Sunspots are darker regions on the surface of the sun that have a relatively cooler temperature of 6,300 degrees Fahrenheit. How many degrees cooler are sunspots?

112. **Jobs.** The Times-Standard reports that over the next year, the credit- and debit-card processing business Humboldt Merchant Services expects to cut 36 of its 80 jobs, but then turn around and hire another 21. How many people will be working for the company then? *Times-Standard 5/6/09*

113. **Wild tigers.** The chart shows the estimated wild tiger population, by region. According to this chart, what is the total wild tiger population worldwide? *Associated Press-Times-Standard 01/24/10 Pressure mounts to save the tiger.*

Region	Tiger population
India, Nepal and Bhutan	1650
China and Russia	450
Bangladesh	250
Sumatra (Indonesia)	400
Malaysia	500
other SE Asia	350

114. Pirate Attacks. The following bar chart tracks pirate attacks off the coast of Somalia.



Source: ICC International Maritime Bureau, AP Times-Standard, 4/15/2009

a) How many pirate attacks were there in 2003, 2004, and 2005 combined?

b) How many pirate attacks were there in 2006, 2007, and 2008 combined?

c) How many more pirate attacks were there in 2008 than in 2007?

115. Emily shows improvement on each successive examination throughout the term. Her exam scores are recorded in the following table.

Exam	Score
Exam #1	48
Exam #2	51
Exam #3	54
Exam #4	59
Exam #5	67
Exam #6	70

a) Create a bar plot for Emily's examination scores. Place the examination numbers on the horizontal axis in the same order shown in the table above.

b) Create a table that shows successive differences in examination scores. Make a line plot of these differences. Between which two exams did Emily show the greatest improvement?

116. Jason shows improvement on each successive examination throughout the term. His exam scores are recorded in the following table.

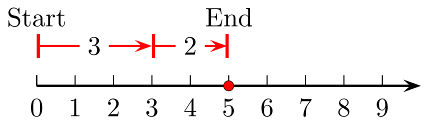
Exam	Score
Exam #1	34
Exam #2	42
Exam #3	45
Exam #4	50
Exam #5	57
Exam #6	62

a) Create a bar plot for Jason's examination scores. Place the examination numbers on the horizontal axis in the same order shown in the table above.

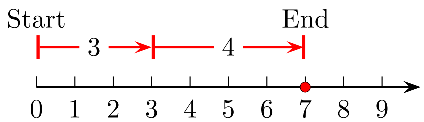
b) Create a table that shows successive differences in examination scores. Make a line plot of these differences. Between which two exams did Jason show the greatest improvement?

Answers

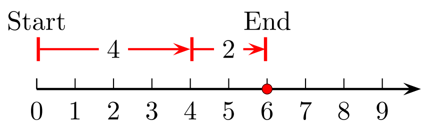
1. $3 + 2 = 5$.



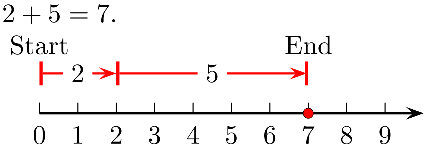
3. $3 + 4 = 7$.



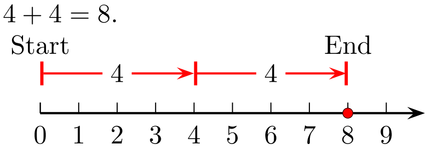
5. $4 + 2 = 6$.



7. $2 + 5 = 7$.



9. $4 + 4 = 8$.



11. Additive identity property of addition.

13. Additive identity property of addition.

15. Associative property of addition

17. Commutative property of addition

19. Associative property of addition

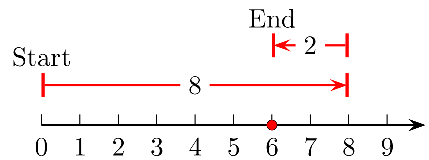
21. Additive identity property of addition.

23. Commutative property of addition

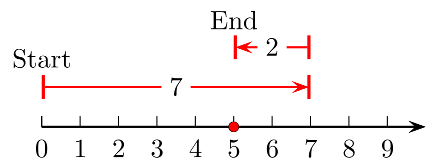
25. Commutative property of addition

27. Associative property of addition

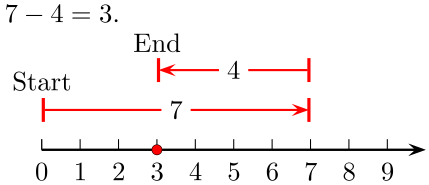
29. $8 - 2 = 6$.



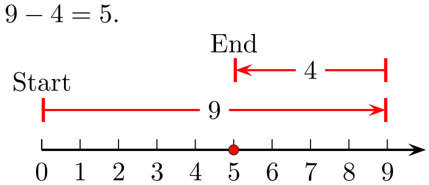
31. $7 - 2 = 5$.



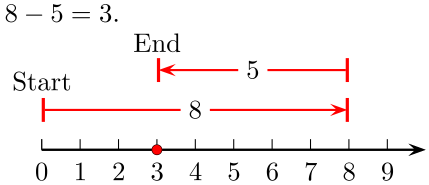
33. $7 - 4 = 3$.



35. $9 - 4 = 5$.



37. $8 - 5 = 3$.



39. 10

41. 29

43. 18

45. 11

47. 10

49. 18

51. $P = 32$ in

53. $P = 34$ in

55. $P = 20$ cm

57. $P = 22$ cm

59. $P = 100$ cm

61. $P = 64$ cm

63. $P = 72$ in

65. $P = 12$ in

67. 8222

69. 1688

71. 1567

73. 1728

75. 9175

77. 1543

79. 1727

81. 2300

83. 1151

85. 723

87. 1232

89. 972

91. 600

93. 3302

95. 728

97. 816

99. 1089

101. 4777

103. 6869

105. $518 million

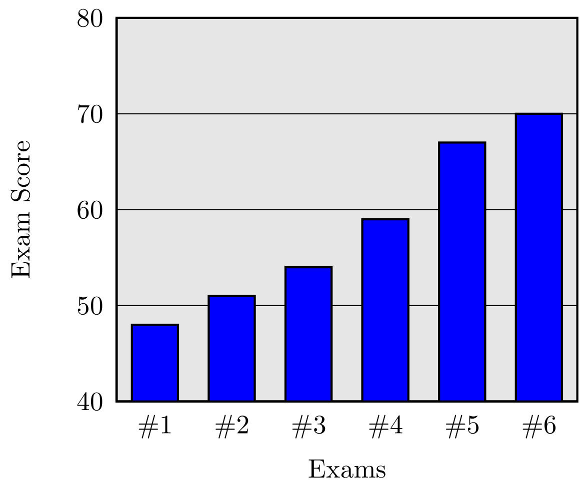
107. $738

109. 5 million kilometers

111. 3,700 degrees Fahrenheit

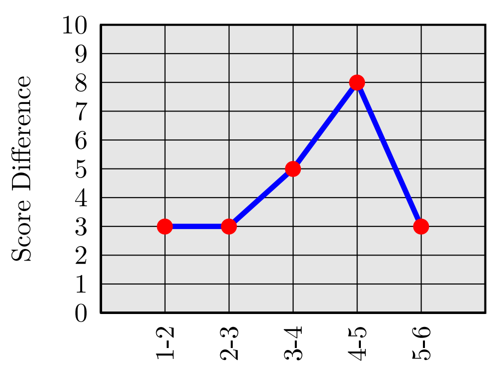
113. 3600

115. a) Bar chart.



b) Line plot of consecutive differences.

The line plot of consecutive examination score differences.



The largest improvement was between Exam #4 and Exam #5, where Emily improved by 8 points.

1.3 Multiplication and Division of Whole Numbers

We begin this section by discussing multiplication of whole numbers. The first order of business is to introduce the various symbols used to indicate multiplication of two whole numbers.

Mathematical symbols that indicate multiplication.

Symbol		Example
$\times$	times symbol	3×4
$\cdot$	dot	$3 \cdot 4$
( )	parentheses	(3)(4) or 3(4) of (3)4

Products and Factors. In the expression $3 \cdot 4$, the whole numbers 3 and 4 are called the **factors** and $3 \cdot 4$ is called the **product**.

The key to understanding multiplication is held in the following statement.

Multiplication is equivalent to repeated addition.

Suppose, for example, that we would like to evaluate the product $3 \cdot 4$. Because multiplication is equivalent to repeated addition, $3 \cdot 4$ is equivalent to adding three fours. That is,

$$3 \cdot 4 = \underbrace{4 + 4 + 4}_{\text{three fours}}$$

Thus, $3 \cdot 4 = 12$. You can visualize the product $3 \cdot 4$ as the sum of three fours on a number line, as shown in Figure 1.6.

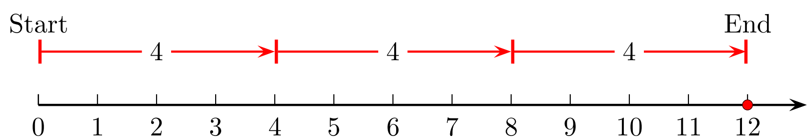


Figure 1.6: Note that $3 \cdot 4 = 4 + 4 + 4$. That is, $3 \cdot 4 = 12$.

Like addition, the order of the factors does not matter.

$$4 \cdot 3 = \underbrace{3 + 3 + 3 + 3}_{\text{four threes}}$$

Thus, $4 \cdot 3 = 12$. Consider the visualization of $4 \cdot 3$ in Figure 1.7.

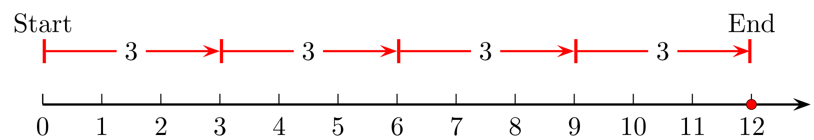


Figure 1.7: Note that $4 \cdot 3 = 3 + 3 + 3 + 3$. That is, $4 \cdot 3 = 12$.

The evidence in Figure 1.6 and Figure 1.7 show us that multiplication is *commutative*. That is,

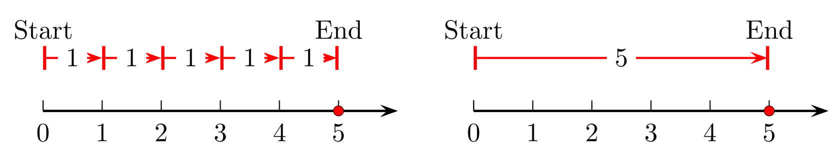
$$3 \cdot 4 = 4 \cdot 3.$$

Commutative Property of Multiplication. If a and b are any whole numbers, then

$$a \cdot b = b \cdot a.$$

The Multiplicative Identity

In Figure 1.8(a), note that five ones equals 5; that is, $5 \cdot 1 = 5$. On the other hand, in Figure 1.8(b), we see that one five equals five; that is, $1 \cdot 5 = 5$.



(a) Note that $5 \cdot 1 = 1 + 1 + 1 + 1 + 1$.

(b) Note that $1 \cdot 5 = 5$.

Figure 1.8: Note that $5 \cdot 1 = 5$ and $1 \cdot 5 = 5$.

Because multiplying a whole number by 1 equals that identical number, the whole number 1 is called the *multiplicative identity*.

The Multiplicative Identity Property. If a is any whole number, then

$$a \cdot 1 = a \qquad \text{and} \qquad 1 \cdot a = a.$$

Multiplication by Zero

Because $3 \cdot 4 = 4 + 4 + 4$, we can say that the product $3 \cdot 4$ represents "3 sets of 4," as depicted in Figure 1.9, where three groups of four boxes are each enveloped in an oval.

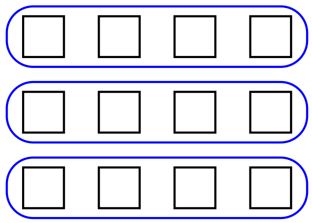


Figure 1.9: Three sets of four: $3 \cdot 4 = 12$.

Therefore, $0 \cdot 4$ would mean zero sets of four. Of course, zero sets of four is zero.

Multiplication by Zero. If a represents any whole number, then

$$a \cdot 0 = 0 \qquad \text{and} \qquad 0 \cdot a = 0.$$

The Associative Property of Multiplication

Like addition, multiplication of whole numbers is associative. Indeed,

$$\begin{aligned} 2 \cdot (3 \cdot 4) &= 2 \cdot 12 \\ &= 24, \end{aligned}$$

and

$$\begin{aligned} (2 \cdot 3) \cdot 4 &= 6 \cdot 4 \\ &= 24. \end{aligned}$$

The Associative Property of Multiplication. If a, b, and c are any whole numbers, then

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

Multiplying Larger Whole Numbers

Much like addition and subtraction of large whole numbers, we will also need to multiply large whole numbers. Again, we hope the algorithm is familiar from previous coursework.

You Try It!

Simplify: $56 \cdot 335$.

EXAMPLE 1. Simplify: $35 \cdot 127$.

Solution. Align the numbers vertically. The order of multiplication does not matter, but we'll put the larger of the two numbers on top of the smaller number. The first step is to multiply 5 times 127. Again, we proceed from right to left. So, 5 times 7 is 35. We write the 5, then carry the 3 to the tens column. Next, 5 times 2 is 10. Add the carry digit 3 to get 13. Write the 3 and carry the 1 to the hundreds column. Finally, 5 times 1 is 5. Add the carry digit to get 6.

$$\begin{array}{r} {\scriptstyle 1\ 3\ \ } \\ 1\,2\,7 \\ \times\ 3\,5 \\ \hline 6\,3\,5 \end{array}$$

The next step is to multiply 3 times 127. However, because 3 is in the tens place, its value is 30, so we actually multiply 30 times 126. This is the same as multiplying 127 by 3 and placing a 0 at the end of the result.

$$\begin{array}{r} {\scriptstyle 2\ \ } \\ 1\,2\,7 \\ \times\ 3\,5 \\ \hline 6\,3\,5 \\ 3\,8\,1\,0 \\ \hline \end{array}$$

After adding the 0, 3 times 7 is 21. We write the 1 and carry the 2 above the 2 in the tens column. Then, 3 times 2 is 6. Add the carry digit 2 to get 8. Finally, 3 times 1 is 1.

All that is left to do is to add the results.

$$\begin{array}{r} 1\,2\,7 \\ \times\ 3\,5 \\ \hline 6\,3\,5 \\ 3\,8\,1\,0 \\ \hline 4\,4\,4\,5 \end{array}$$

Thus, $35 \cdot 127 = 4,445$.

Alternate Format. It does not hurt to omit the trailing zero in the second step of the multiplication, where we multiply 3 times 127. The result would look like this:

$$\begin{array}{r} 1\,2\,7 \\ \times\ 3\,5 \\ \hline 6\,3\,5 \\ 3\,8\,1 \\ \hline 4\,4\,4\,5 \end{array}$$

In this format, the zero is understood, so it is not necessary to have it physically present. The idea is that with each multiplication by a new digit, we indent the product one space from the right.

Answer: $18,760$

□

Division of Whole Numbers

We now turn to the topic of division of whole numbers. We first introduce the various symbols used to indicate division of whole numbers.

Mathematical symbols that indicate division.

Symbol		Example
$\div$	division symbol	$12 \div 4$
–	fraction bar	$\frac{12}{4}$
$\overline{)}$	division bar	$4\overline{)12}$

Note that each of the following say the same thing; that is, "12 divided by 4 is 3."

$$12 \div 4 = 3 \qquad \text{or} \qquad \frac{12}{4} = 3 \qquad \text{or} \qquad \begin{array}{r} 3 \\ 4\overline{)12} \end{array}$$

Quotients, Dividends, and Divisors. In the statement

$$\begin{array}{r} 3 \\ 4\overline{)12} \end{array}$$

the whole number 12 is called the *dividend*, the whole number 4 is called the *divisor*, and the whole number 3 is called the *quotient*. Note that this division bar notation is equivalent to

$$12 \div 4 = 3 \qquad \text{and} \qquad \frac{12}{4} = 3.$$

The expression a/b means "a divided by b," but this construct is also called a *fraction*.

Fraction. The expression

$$\frac{a}{b}$$

is called a *fraction*. The number a on top is called the *numerator* of the fraction; the number b on the bottom is called the *denominator* of the fraction.

The key to understanding division of whole numbers is contained in the following statement.

Division is equivalent to repeated subtraction.

Suppose for example, that we would like to divide the whole number 12 by the whole number 4. This is equivalent to asking the question "how many fours can we subtract from 12?" This can be visualized in a number line diagram, such as the one in Figure 1.10.

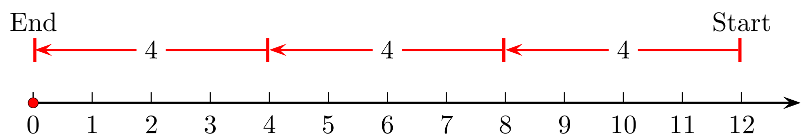


Figure 1.10: Division is repeated subtraction.

In Figure 1.10, note that we if we subtract three fours from twelve, the result is zero. In symbols,

$$12 - \underbrace{4 - 4 - 4}_{\text{three fours}} = 0.$$

Equivalently, we can also ask "How many groups of four are there in 12," and arrange our work as shown in Figure 1.11, where we can see that in an array of twelve objects, we can circle three groups of four ; i.e., $12 \div 4 = 3$.

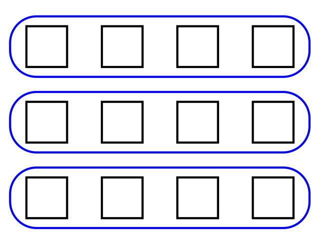


Figure 1.11: There are three groups of four in twelve.

In Figure 1.10 and Figure 1.11, note that the division (repeated subtraction) leaves no remainder. This is not always the case.

You Try It!

Use both the number line approach and the array of boxes approach to divide 12 by 5.

EXAMPLE 2. Divide 7 by 3.

Solution. In Figure 1.12, we see that we can subtract two threes from seven, leaving a remainder of one.

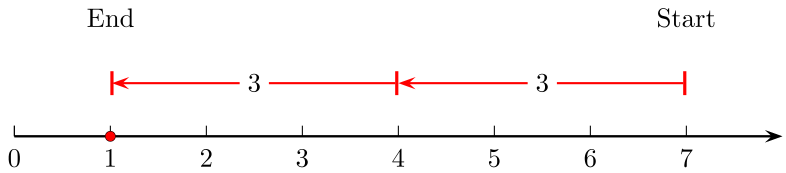


Figure 1.12: Division with a remainder.

Alternatively, in an array of seven objects, we can circle two groups of three, leaving a remainder of one.

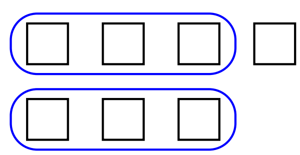


Figure 1.13: Dividing seven by three leaves a remainder of one.

Both Figure 1.12 and Figure1.13 show that there are two groups of three in seven, with one left over. We say "Seven divided by three is two, with a remainder of one."

□

Division is not Commutative

When dividing whole numbers, the order matters. For example,

$$12 \div 4 = 3,$$

but $4 \div 12$ is not even a whole number. Thus, if a and b are whole numbers, then $a \div b$ does **not** have to be the same as $b \div a$.

Division is not Associative

When you divide three numbers, the order in which they are grouped will usually affect the answer. For example,

$$\begin{aligned}(48 \div 8) \div 2 &= 6 \div 2 \\ &= 3,\end{aligned}$$

but

$$\begin{aligned} 48 \div (8 \div 2) &= 48 \div 4 \\ &= 12. \end{aligned}$$

Thus, if a, b, and c are whole numbers, $(a \div b) \div c$ does **not** have to be the same as $a \div (b \div c)$.

Division by Zero is Undefined

Suppose that we are asked to divide six by zero; that is, we are asked to calculate $6 \div 0$. In Figure 1.14, we have an array of six objects.

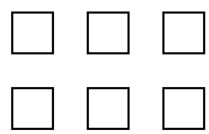


Figure 1.14: How many groups of zero do you see?

Now, to divide six by zero, we must answer the question "How many groups of zero can we circle in Figure 1.14?" Some thought will provide the answer: This is a meaningless request! It makes absolutely no sense to ask how many groups of zero can be circled in the array of six objects in Figure 1.14.

Division by Zero. Division by zero is **undefined**. Each of the expressions

$$6 \div 0 \qquad \text{and} \qquad \frac{6}{0} \qquad \text{and} \qquad 0\overline{)6}$$

is **undefined**.

On the other hand, it make sense to ask "What is zero divided by six?" If we create an array of zero objects, then ask how many groups of six we can circle, the answer is "zero groups of six." That is, zero divided by six is zero.

$$0 \div 6 = 0 \qquad \text{and} \qquad \frac{0}{6} = 0 \qquad \text{and} \qquad \begin{array}{r} 0 \\ 6\overline{)0}. \end{array}$$

Dividing Larger Whole Numbers

We'll now provide a quick review of division of larger whole numbers, using an algorithm that is commonly called *long division*. This is not meant to be a thorough discussion, but a cursory one. We're counting on the fact that our readers have encountered this algorithm in previous courses and are familiar with the process.

You Try It!

Divide: 980/35

EXAMPLE 3. Simplify: 575/23.

Solution. We begin by estimating how many times 23 will divide into 57, guessing 1. We put the 1 in the quotient above the 7, multiply 1 times 23, place the answer underneath 57, then subtract.

$$\begin{array}{r} 1 \\ 23\overline{)575} \\ \underline{23} \\ 34 \end{array}$$

Because the remainder is larger than the divisor, our estimate is too small. We try again with an estimate of 2.

$$\begin{array}{r} 2 \\ 23\overline{)575} \\ \underline{46} \\ 11 \end{array}$$

That's the algorithm. Divide, multiply, then subtract. You may continue only when the remainder is smaller than the divisor.

To continue, bring down the 5, estimate that 115 divided by 23 is 5, then multiply 5 times the divisor and subtract.

$$\begin{array}{r} 25 \\ 23\overline{)575} \\ \underline{46} \\ 115 \\ \underline{115} \\ 0 \end{array}$$

Because the remainder is zero, $575/23 = 25$.

Answer: 28

Application — Counting Rectangular Arrays

Consider the rectangular array of stars in Figure 1.15.

To count the number of stars in the array, we could use brute force, counting each star in the array one at a time, for a total of 20 stars. However, as we have four rows of five stars each, it is much faster to multiply: $4 \cdot 5 = 20$ stars.

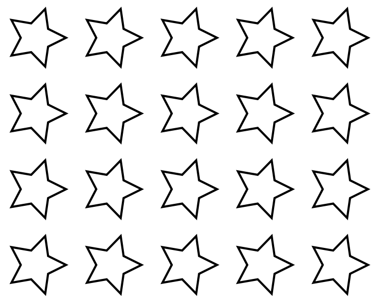


Figure 1.15: Four rows and five columns.

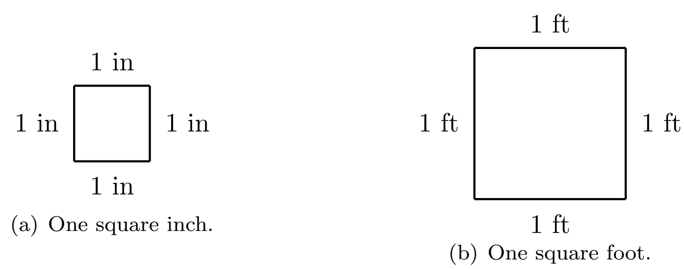


Figure 1.16: Measures of area are in square units.

Application — Area

In Figure 1.16(a), pictured is one square inch ($1\,\text{in}^2$), a square with one inch on each side. In Figure 1.16(b), pictured is one square foot ($1\,\text{ft}^2$), a square with one foot on each side. Both of these squares are *measures of area*.

Now, consider the rectangle shown in Figure 1.17. The length of this rectangle is four inches (4 in) and the width is three inches (3 in).

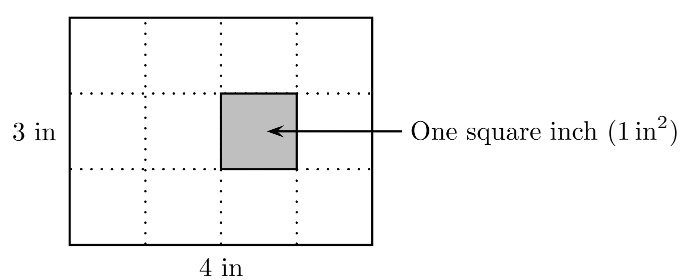
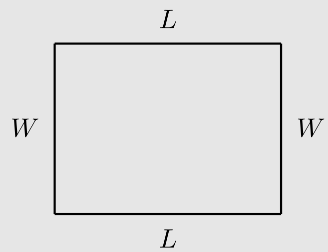


Figure 1.17: A rectangle with length 4 inches and width 3 inches.

To find the area of the figure, we can count the individual units of area that make up the area of the rectangle, twelve square inches ($12\,\text{in}^2$) in all. However, as we did in counting the stars in the array in Figure 1.15, it is much faster to note that we have three rows of four square inches. Hence, it is much faster to multiply the number of squares in each row by the number of squares in each column: $4 \cdot 3 = 12$ square inches.

The argument presented above leads to the following rule for finding the area of a rectangle.

Area of a Rectangle. Let L and W represent the length and width of a rectangle, respectively.



To find the area of the rectangle, calculate the product of the length and width. That is, if A represents the area of the rectangle, then the area of the rectangle is given by the formula

$$A = LW.$$

You Try It!

EXAMPLE 4. A rectangle has width 5 feet and length 12 feet. Find the area of the rectangle.

Solution. Substitute $L = 12\,\text{ft}$ and $W = 5\,\text{ft}$ into the area formula.

$$\begin{aligned} A &= LW \\ &= (12\,\text{ft})(5\,\text{ft}) \\ &= 60\,\text{ft}^2 \end{aligned}$$

Hence, the area of the rectangle is 60 square feet.

A rectangle has width 17 inches and length 33 inches. Find the area of the rectangle.

Answer: 561 square inches.

□

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-4 use number line diagrams as shown in Figure 1.6 to depict the multiplication.

1. $2 \cdot 4$.

2. $3 \cdot 4$.

3. $4 \cdot 2$.

4. $4 \cdot 3$.

In Exercises 5-16, state the property of multiplication depicted by the given identity.

5. $9 \cdot 8 = 8 \cdot 9$

6. $5 \cdot 8 = 8 \cdot 5$

7. $8 \cdot (5 \cdot 6) = (8 \cdot 5) \cdot 6$

8. $4 \cdot (6 \cdot 5) = (4 \cdot 6) \cdot 5$

9. $6 \cdot 2 = 2 \cdot 6$

10. $8 \cdot 7 = 7 \cdot 8$

11. $3 \cdot (5 \cdot 9) = (3 \cdot 5) \cdot 9$

12. $8 \cdot (6 \cdot 4) = (8 \cdot 6) \cdot 4$

13. $21 \cdot 1 = 21$

14. $39 \cdot 1 = 39$

15. $13 \cdot 1 = 13$

16. $44 \cdot 1 = 44$

In Exercises 17-28, multiply the given numbers.

17. $78 \cdot 3$

18. $58 \cdot 7$

19. $907 \cdot 6$

20. $434 \cdot 80$

21. $128 \cdot 30$

22. $454 \cdot 90$

23. $799 \cdot 60$

24. $907 \cdot 20$

25. $14 \cdot 70$

26. $94 \cdot 90$

27. $34 \cdot 90$

28. $87 \cdot 20$

In Exercises 29-40, multiply the given numbers.

29. $237 \cdot 54$

30. $893 \cdot 94$

31. $691 \cdot 12$

32. $823 \cdot 77$

33. $955 \cdot 89$

34. $714 \cdot 41$

35. $266 \cdot 61$

36. $366 \cdot 31$

37. $365 \cdot 73$

38. $291 \cdot 47$

39. $955 \cdot 57$

40. $199 \cdot 33$

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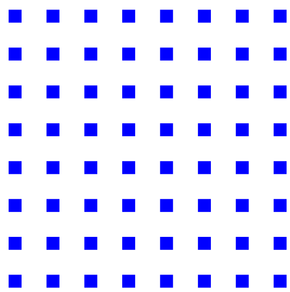
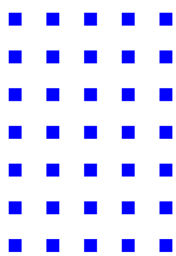
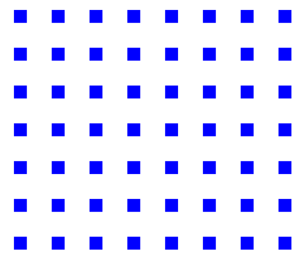
41. Count the number of objects in the array.



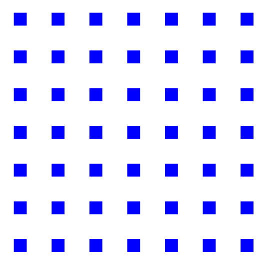
42. Count the number of objects in the array.



43. Count the number of objects in the array.



44. Count the number of objects in the array.



In Exercises 45-48, find the area of the rectangle having the given length and width.

45. $L = 50$ in, $W = 25$ in

46. $L = 48$ in, $W = 24$ in

47. $L = 47$ in, $W = 13$ in

48. $L = 19$ in, $W = 10$ in

In Exercises 49-52, find the perimeter of the rectangle having the given length and width.

49. $L = 25$ in, $W = 16$ in

50. $L = 34$ in, $W = 18$ in

51. $L = 30$ in, $W = 28$ in

52. $L = 41$ in, $W = 25$ in

---

53. A set of beads costs 50 cents per dozen. What is the cost (in dollars) of 19 dozen sets of beads?

54. A set of beads costs 60 cents per dozen. What is the cost (in dollars) of 7 dozen sets of beads?

55. If a math tutor worked for 47 hours and was paid $15 each hour, how much money would she have made?

56. If a math tutor worked for 46 hours and was paid $11 each hour, how much money would he have made?

57. There are 12 eggs in one dozen, and 12 dozen in one gross. How many eggs are in a shipment of 24 gross?

58. There are 12 eggs in one dozen, and 12 dozen in one gross. How many eggs are in a shipment of 11 gross?

59. If bricks weigh 4 kilograms each, what is the weight (in kilograms) of 5000 bricks?

60. If bricks weigh 4 pounds each, what is the weight (in pounds) of 2000 bricks?

In Exercises 61-68, which of the following four expressions differs from the remaining three?

61. $\frac{30}{5}$, $30 \div 5$, $5\overline{)30}$, $5 \div 30$

62. $\frac{12}{2}$, $12 \div 2$, $2\overline{)12}$, $2 \div 12$

63. $\frac{8}{2}$, $8 \div 2$, $2\overline{)8}$, $8\overline{)2}$

64. $\frac{8}{4}$, $8 \div 4$, $4\overline{)8}$, $8\overline{)4}$

65. $2\overline{)14}$, $14\overline{)2}$, $\frac{14}{2}$, $14 \div 2$

66. $9\overline{)54}$, $54\overline{)9}$, $\frac{54}{9}$, $54 \div 9$

67. $3\overline{)24}$, $3 \div 24$, $\frac{24}{3}$, $24 \div 3$

68. $3\overline{)15}$, $3 \div 15$, $\frac{15}{3}$, $15 \div 3$

In Exercises 69-82, simplify the given expression. If the answer doesn't exist or is undefined, write "undefined".

69. $0 \div 11$

70. $0 \div 5$

71. $17 \div 0$

72. $24 \div 0$

73. $10 \cdot 0$

74. $20 \cdot 0$

75. $\frac{7}{0}$

76. $\frac{23}{0}$

77. $16\overline{)0}$

78. $25\overline{)0}$

79. $\frac{0}{24}$

80. $\frac{0}{22}$

81. $0\overline{)0}$

82. $0 \div 0$

In Exercises 83-94, divide the given numbers.

83. $\dfrac{2816}{44}$

84. $\dfrac{1998}{37}$

85. $\dfrac{2241}{83}$

86. $\dfrac{2716}{97}$

87. $\dfrac{3212}{73}$

88. $\dfrac{1326}{17}$

89. $\dfrac{8722}{98}$

90. $\dfrac{1547}{91}$

91. $\dfrac{1440}{96}$

92. $\dfrac{2079}{27}$

93. $\dfrac{8075}{85}$

94. $\dfrac{1587}{23}$

In Exercises 95-106, divide the given numbers.

95. $\dfrac{17756}{92}$

96. $\dfrac{46904}{82}$

97. $\dfrac{11951}{19}$

98. $\dfrac{22304}{41}$

99. $\dfrac{18048}{32}$

100. $\dfrac{59986}{89}$

101. $\dfrac{29047}{31}$

102. $\dfrac{33264}{36}$

103. $\dfrac{22578}{53}$

104. $\dfrac{18952}{46}$

105. $\dfrac{12894}{14}$

106. $\dfrac{18830}{35}$

---

107. A concrete sidewalk is laid in square blocks that measure 6 feet on each side. How many blocks will there be in a walk that is 132 feet long?

108. A concrete sidewalk is laid in square blocks that measure 5 feet on each side. How many blocks will there be in a walk that is 180 feet long?

109. One boat to the island can take 5 people. How many trips will the boat have to take in order to ferry 38 people to the island? (Hint: Round up your answer.)

110. One boat to the island can take 4 people. How many trips will the boat have to take in order to ferry 46 people to the island? (Hint: Round up your answer.)

111. If street lights are placed at most 145 feet apart, how many street lights will be needed for a street that is 4 miles long, assuming that there are lights at each end of the street? (Note: 1 mile = 5280 feet.)

112. If street lights are placed at most 70 feet apart, how many street lights will be needed for a street that is 3 miles long, assuming that there are lights at each end of the street? (Note: 1 mile = 5280 feet.)

113. A concrete sidewalk is laid in square blocks that measure 4 feet on each side. How many blocks will there be in a walk that is 292 feet long?

114. A concrete sidewalk is laid in square blocks that measure 5 feet on each side. How many blocks will there be in a walk that is 445 feet long?

115. One boat to the island can take 3 people. How many trips will the boat have to take in order to ferry 32 people to the island? (Hint: Round up your answer.)

116. One boat to the island can take 4 people. How many trips will the boat have to take in order to ferry 37 people to the island? (Hint: Round up your answer.)

117. If street lights are placed at most 105 feet apart, how many street lights will be needed for a street that is 2 miles long, assuming that there are lights at each end of the street? (Note: 1 mile = 5280 feet.)

118. If street lights are placed at most 105 feet apart, how many street lights will be needed for a street that is 3 miles long, assuming that there are lights at each end of the street? (Note: 1 mile = 5280 feet.)

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119. **Writing articles.** Eli writes an average of 4 articles a day, five days a week, to support product sales. How many articles does Eli write in one week?

120. **Machine gun.** A 0.50-caliber anti-aircraft machine gun can fire 800 rounds each minute. How many rounds could fire in three minutes? *Associated Press Times-Standard 4/15/09*

121. **Laps.** The swimming pool at CalCourts is 25 yards long. If one lap is up and back again, how many yards has Wendell swam doing 27 laps?

122. **Refrigerator wattage.** A conventional refrigerator will run about 12 hours each day can use 150 Watts of power each hour. How many Watts of power will a refrigerator use over the day?

123. **Horse hay.** A full-grown horse should eat a minimum of 12 pounds of hay each day and may eat much more depending on their weight. How many pounds minimum would a horse eat over a year?

124. **College costs.** After a $662 hike in fees, Califormia residents who want to attend the University of California as an undergraduate should expect to pay $8,700 in for the upcoming academic year 2009-2010. If the cost were to remain the same for the next several years, how much should a student expect to pay for a four-year degree program at a UC school?

125. **Non-resident costs.** Nonresident undergraduates who want to attend a University of California college should expect to pay about $22,000 for the upcoming academic year. Assuming costs remain the same, what can a four-year degree cost?

126. **Student tax.** The mayer of Providence, Rhode Island wants to tax its 25,000 Brown University students $150 each to contribute to tax receipts saying students should pay for the resources they use just like the town residents. How many dollars would the mayer generate?

127. **New iceberg.** A new iceberg, shaved off a glacier after a collision with another iceberg, measures about 48 miles long and 28 miles wide. What's the approximate area of the new iceberg? *Associated Press-Times-Standard 02/27/10 2 Huge icebergs set loose off Antarctica's coast.*

128. **Solar panels.** One of the solar panels on the International Space Station is 34 meters long and 11 meters wide. If there are eight of these, what's the total area for solar collection?

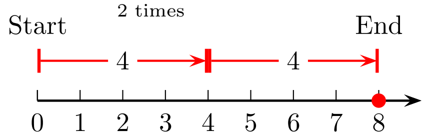
129. **Sidewalk.** A concrete sidewalk is to be 80 foot long and 4 foot wide. How much will it cost to lay the sidewalk at $8 per square foot?

130. **Hay bales.** An average bale of hay weighs about 60 pounds. If a horse eats 12 pounds of hay a day, how many days will one bale feed a horse?

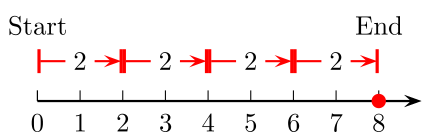
131. **Sunspots.** Sunspots, where the sun's magnetic field is much higher, usually occur in pairs. If the total count of sunspots is 72, how many pairs of sunspots are there?

Answers

1. $2 \cdot 4 = \underbrace{4 + 4}_{\text{2 times}} = 8$



3. $4 \cdot 2 = \underbrace{2 + 2 + 2 + 2}_{\text{4 times}} = 8$



5. Commutative property of multiplication

7. Associative property of multiplication

9. Commutative property of multiplication

11. Associative property of multiplication

13. Multiplicative identity property

15. Multiplicative identity property

17. 234

19. 5442

21. 3840

23. 47940

25. 980

27. 3060

29. 12798

31. 8292

33. 84995

35. 16226

37. 26645

39. 54435

41. 64

43. 56

45. $1250\,\mathrm{in}^2$

47. $611\,\mathrm{in}^2$

49. 82 in

51. 116 in

53. 9.50

55. 705

57. 3456

59. 20000

61. $5 \div 30$

63. $8\overline{)2}$

65. $14\overline{)2}$

67. $3 \div 24$

69. 0

71. Undefined

73. 0

75. Undefined

77. 0

79. 0

81. Undefined

83. 64

85. 27

87. 44

89. 89

91. 15

93. 95

95. 193

97. 629

99. 564

101. 937

103. 426

105. 921

107. 22

109. 8

111. 147

113. 73

115. 11

117. 102

119. 20 articles

121. 1350 yards

123. 4380 pounds of hay

125. $88,000

127. $1344\,\mathrm{mi}^2$

129. $2,560

131. 36

1.4 Prime Factorization

In the statement $3 \cdot 4 = 12$, the number 12 is called the *product*, while 3 and 4 are called *factors*.

You Try It!

EXAMPLE 1. Find all whole number factors of 18.

Solution. We need to find all whole number pairs whose product equals 18. The following pairs come to mind.

$$1 \cdot 18 = 18 \quad \text{and} \quad 2 \cdot 9 = 18 \quad \text{and} \quad 3 \cdot 6 = 18.$$

Hence, the factors of 18 are (in order) 1, 2, 3, 6, 9, and 18.

Find all whole number factors of 21.

Answer: 1, 3, 7, and 21.

Divisibility

In Example 1, we saw $3 \cdot 6 = 18$, making 3 and 6 factors of 18. Because division is the inverse of multiplication, that is, divison by a number undoes the multiplication of that number, this immediately provides

$$18 \div 6 = 3 \quad \text{and} \quad 18 \div 3 = 6.$$

That is, 18 is divisible by 3 and 18 is divisible by 6. When we say that 18 is divisible by 3, we mean that when 18 is divided by 3, there is a zero remainder.

Divisible. Let a and b be whole numbers. Then a is *divisible* by b if and only if the remainder is **zero** when a is divided by b. In this case, we say that "b is a *divisor* of a."

You Try It!

EXAMPLE 2. Find all whole number divisors of 18.

Solution. In Example 1, we saw that $3 \cdot 6 = 18$. Therefore, 18 is divisible by both 3 and 6 ($18 \div 3 = 6$ and $18 \div 6 = 3$). Hence, when 18 is divided by 3 or 6, the remainder is zero. Therefore, 3 and 6 are divisors of 18. Noting the other products in Example 1, the complete list of divisors of 18 is 1, 2, 3, 6, 9, and 18.

Find all whole number divisors of 21.

Answer: 1, 3, 7, and 21.

Example 1 and Example 2 show that when working with whole numbers, the words *factor* and *divisor* are interchangeable.

Factors and Divisors. If

$$c = a \cdot b,$$

then a and b are called *factors* of c. Both a and b are also called *divisors* of c.

Divisibility Tests

There are a number of very useful divisibility tests.

Divisible by 2. If a whole number ends in 0, 2, 4, 6, or 8, then the number is called an **even** number and is divisible by 2. Examples of even numbers are 238 and 1,246 ($238 \div 2 = 119$ and $1,246 \div 2 = 623$). A number that is **not** even is called an **odd** number. Examples of odd numbers are 113 and 2,339.

Divisible by 3. If the sum of the digits of a whole number is divisible by 3, then the number itself is divisible by 3. An example is 141. The sum of the digits is $1 + 4 + 1 = 6$, which is divisible by 3. Therefore, 141 is also divisible by 3 ($141 \div 3 = 47$).

Divisible by 4. If the number represented by the last two digits of a whole number is divisible by 4, then the number itself is divisible by 4. An example is 11,524. The last two digits represent 24, which is divisible by 4 ($24 \div 4 = 6$). Therefore, 11,524 is divisible by 4 ($11,524 \div 4 = 2,881$).

Divisible by 5. If a whole number ends in a zero or a 5, then the number is divisible by 5. Examples are 715 and 120 ($715 \div 5 = 143$ and $120 \div 5 = 24$).

Divisible by 6. If a whole number is divisible by 2 and by 3, then it is divisible by 6. An example is 738. First, 738 is even and divisible by 2. Second, 7+3+8=18, which is divisible by 3. Hence, 738 is divisible by 3. Because 738 is divisible by both 2 and 3, it is divisible by 6 ($738 \div 6 = 123$).

Divisible by 8. If the number represented by the last three digits of a whole number is divisible by 8, then the number itself is divisible by 8. An example is 73,024. The last three digits represent the number 24, which is divisible by 8 ($24 \div 8 = 3$). Thus, 73,024 is also divisible by 8 ($73,024 \div 8 = 9,128$).

Divisible by 9. If the sum of the digits of a whole number is divisible by 9, then the number itself is divisible by 9. An example is 117. The sum of the digits is $1 + 1 + 7 = 9$, which is divisible by 9. Hence, 117 is divisible by 9 ($117 \div 9 = 13$).

Prime Numbers

We begin with the definition of a *prime number*.

Prime Number. A whole number (other than 1) is a *prime number* if its only factors (divisors) are 1 and itself. Equivalently, a number is prime if and only if it has exactly two factors (divisors).

You Try It!

EXAMPLE 3. Which of the whole numbers 12, 13, 21, and 37 are prime numbers?

Solution.

- The factors (divisors) of 12 are 1, 2, 3, 4, 6, and 12. Hence, 12 is **not** a prime number.
- The factors (divisors) of 13 are 1 and 13. Because its only divisors are 1 and itself, 13 **is** a prime number.
- The factors (divisors) of 21 are 1, 3, 7, and 21. Hence, 21 is **not** a prime number.
- The factors (divisors) of 37 are 1 and 37. Because its only divisors are 1 and itself, 37 **is** a prime number.

□

Which of the whole numbers 15, 23, 51, and 59 are prime numbers?

Answer: 23 and 59.

You Try It!

EXAMPLE 4. List all the prime numbers less than 20.

Solution. The prime numbers less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19.

□

List all the prime numbers less than 100.

Composite Numbers. If a whole number is not a prime number, then it is called a *composite number*.

You Try It!

Is the whole number 2,571 prime or composite?

Answer: Composite.

EXAMPLE 5. Is the whole number 1,179 prime or composite?

Solution. Note that $1 + 1 + 7 + 9 = 18$, which is divisible by both 3 and 9. Hence, 3 and 9 are both divisors of 1,179. Therefore, 1,179 is a composite number.

□

Factor Trees

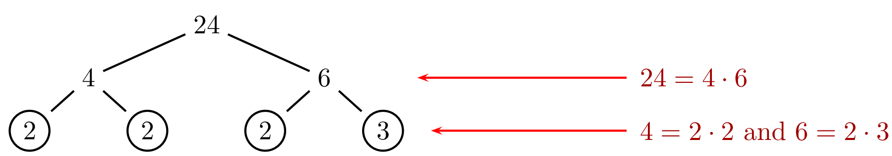
We will now learn how to express a composite number as a unique product of prime numbers. The most popular device for accomplishing this goal is the *factor tree.*

You Try It!

Express 36 as a product of prime factors.

EXAMPLE 6. Express 24 as a product of prime factors.

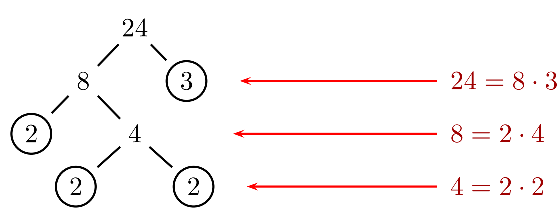
Solution. We use a factor tree to break 24 down into a product of primes.



At each level of the tree, break the current number into a product of two factors. The process is complete when all of the "circled leaves" at the bottom of the tree are prime numbers. Arranging the factors in the "circled leaves" in order,

$$24 = 2 \cdot 2 \cdot 2 \cdot 3.$$

The final answer does not depend on product choices made at each level of the tree. Here is another approach.



The final answer is found by including all of the factors from the "circled leaves" at the end of each branch of the tree, which yields the same result, namely $24 = 2 \cdot 2 \cdot 2 \cdot 3$.

Alternate Approach. Some favor repeatedly dividing by 2 until the result is no longer divisible by 2. Then try repeatedly dividing by the next prime until the result is no longer divisible by that prime. The process terminates when the last resulting quotient is equal to the number 1.

2	24	
2	12	← $24 \div 2 = 12$
2	6	← $12 \div 2 = 6$
3	3	← $6 \div 2 = 3$
	1	← $3 \div 3 = 1$

The first column reveals the prime factorization; i.e., $24 = 2 \cdot 2 \cdot 2 \cdot 3$.

Answer: $2 \cdot 2 \cdot 3 \cdot 3$.

The fact that the alternate approach in Example 6 yielded the same result is significant.

Unique Factorization Theorem. Every whole number can be **uniquely** factored as a product of primes.

This result guarantees that if the prime factors are ordered from smallest to largest, everyone will get the same result when breaking a number into a product of prime factors.

Exponents

We begin with the definition of an exponential expression.

Exponents. The expression a^m is defined to mean

$$a^m = \underbrace{a \cdot a \cdot \ldots \cdot a}_{m \text{ times}}.$$

The number a is called the *base* of the exponential expression and the number m is called the *exponent*. The exponent m tells us to repeat the base a as a factor m times.

You Try It!

Evaluate: 3^5.

EXAMPLE 7. Evaluate 2^5, 3^3 and 5^2.

Movie

Solution.

- In the case of 2^5, we have

$$\begin{aligned} 2^5 &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \\ &= 32. \end{aligned}$$

- In the case of 3^3, we have

$$\begin{aligned} 3^3 &= 3 \cdot 3 \cdot 3 \\ &= 27. \end{aligned}$$

- In the case of 5^2, we have

$$\begin{aligned} 5^2 &= 5 \cdot 5 \\ &= 25. \end{aligned}$$

Answer: 243.

□

You Try It!

Prime factor 54.

EXAMPLE 8. Express the solution to Example 6 in compact form using exponents.

Solution. In Example 6, we determined the prime factorization of 24.

$$24 = 2 \cdot 2 \cdot 2 \cdot 3$$

Because $2 \cdot 2 \cdot 2 = 2^3$, we can write this more compactly.

$$24 = 2^3 \cdot 3$$

Answer: $2 \cdot 3 \cdot 3 \cdot 3$.

□

You Try It!

Evaluate: $3^3 \cdot 5^2$.

EXAMPLE 9. Evaluate the expression $2^3 \cdot 3^2 \cdot 5^2$.

Solution. First raise each factor to the given exponent, then perform the multiplication in order (left to right).

$$\begin{aligned} 2^3 \cdot 3^2 \cdot 5^2 &= 8 \cdot 9 \cdot 25 \\ &= 72 \cdot 25 \\ &= 1800 \end{aligned}$$

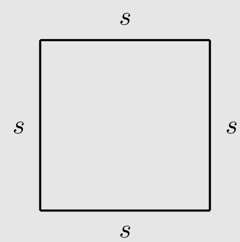
Answer: 675

□

Application

A square is a rectangle with four equal sides.

Area of a Square. Let s represent the length of each side of a square.



Because a square is also a rectangle, we can find the area of the square by multiplying its length and width. However, in this case, the length and width both equal s, so $A = (s)(s) = s^2$. Hence, the formula for the area of a square is

$$A = s^2.$$

You Try It!

EXAMPLE 10. The edge of a square is 13 centimeters. Find the area of the square.

Solution. Substitute $s = 13\,\text{cm}$ into the area formula.

$$\begin{aligned} A &= s^2 \\ &= (13\,\text{cm})^2 \\ &= (13\,\text{cm})(13\,\text{cm}) \\ &= 169\,\text{cm}^2 \end{aligned}$$

Hence, the area of the square is $169\,\text{cm}^2$; i.e., 169 square centimeters.

The edge of a square is 15 meters. Find the area of the square.

Answer: 225 square meters.

□

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-12, find all divisors of the given number.

1. 30

2. 19

3. 83

4. 51

5. 91

6. 49

7. 75

8. 67

9. 64

10. 87

11. 14

12. 89

In Exercises 13-20, which of the following numbers is **not** divisible by 2?

13. 117, 120, 342, 230

14. 310, 157, 462, 160

15. 30, 22, 16, 13

16. 382, 570, 193, 196

17. 105, 206, 108, 306

18. 60, 26, 23, 42

19. 84, 34, 31, 58

20. 66, 122, 180, 63

In Exercises 21-28, which of the following numbers is **not** divisible by 3?

21. 561, 364, 846, 564

22. 711, 850, 633, 717

23. 186, 804, 315, 550

24. 783, 909, 504, 895

25. 789, 820, 414, 663

26. 325, 501, 945, 381

27. 600, 150, 330, 493

28. 396, 181, 351, 606

In Exercises 29-36, which of the following numbers is **not** divisible by 4?

29. 3797, 7648, 9944, 4048

30. 1012, 9928, 7177, 1592

31. 9336, 9701, 4184, 2460

32. 2716, 1685, 2260, 9788

33. 9816, 7517, 8332, 7408

34. 1788, 8157, 7368, 4900

35. 1916, 1244, 7312, 7033

36. 7740, 5844, 2545, 9368

In Exercises 37-44, which of the following numbers is **not** divisible by 5?

37. 8920, 4120, 5285, 9896

38. 3525, 7040, 2185, 2442

39. 8758, 3005, 8915, 3695

40. 3340, 1540, 2485, 2543

41. 2363, 5235, 4145, 4240

42. 9030, 8000, 5445, 1238

43. 1269, 5550, 4065, 5165

44. 7871, 9595, 3745, 4480

In Exercises 45-52, which of the following numbers is **not** divisible by 6?

45. 328, 372, 990, 528

46. 720, 288, 148, 966

47. 744, 174, 924, 538

48. 858, 964, 930, 330

49. 586, 234, 636, 474

50. 618, 372, 262, 558

51. 702, 168, 678, 658

52. 780, 336, 742, 312

In Exercises 53-60, which of the following numbers is **not** divisible by 8?

53. 1792, 8216, 2640, 5418

54. 2168, 2826, 1104, 2816

55. 8506, 3208, 9016, 2208

56. 2626, 5016, 1392, 1736

57. 4712, 3192, 2594, 7640

58. 9050, 9808, 8408, 7280

59. 9808, 1232, 7850, 7912

60. 3312, 1736, 9338, 3912

In Exercises 61-68, which of the following numbers is **not** divisible by 9?

61. 477, 297, 216, 991

62. 153, 981, 909, 919

63. 153, 234, 937, 675

64. 343, 756, 927, 891

65. 216, 783, 594, 928

66. 504, 279, 307, 432

67. 423, 801, 676, 936

68. 396, 684, 567, 388

In Exercises 69-80, identify the given number as prime, composite, or neither.

69. 19

70. 95

71. 41

72. 88

73. 27

74. 61

75. 91

76. 72

77. 21

78. 65

79. 23

80. 36

In Exercises 81-98, find the prime factorization of the natural number.

81. 224

82. 320

83. 108

84. 96

85. 243

86. 324

87. 160

88. 252

89. 32

90. 128

91. 360

92. 72

93. 144

94. 64

95. 48

96. 200

97. 216

98. 392

In Exercises 99-110, compute the exact value of the given exponential expression.

99. $5^2 \cdot 4^1$

100. $2^3 \cdot 4^1$

101. 0^1

102. 1^3

103. $3^3 \cdot 0^2$

104. $3^3 \cdot 2^2$

105. 4^1

106. 5^2

107. 4^3

108. 4^2

109. $3^3 \cdot 1^2$

110. $5^2 \cdot 2^3$

In Exercises 111-114, find the area of the square with the given side.

111. 28 inches

112. 31 inches

113. 22 inches

114. 13 inches

Create factor trees for each number in Exercises 115-122. Write the prime factorization for each number in compact form, using exponents.

115. 12

116. 18

117. 105

118. 70

119. 56

120. 56

121. 72

122. 270

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123. **Sieve of Eratosthenes.** This exercise introduces the *Sieve of Eratosthenes*, an ancient algorithm for finding the primes less than a certain number n, first created by the Greek mathematician Eratosthenes. Consider the grid of integers from 2 through 100.

2	3	4	5	6	7	8	9	10	11
12	13	14	15	16	17	18	19	20	21
22	23	24	25	26	27	28	29	30	31
32	33	34	35	36	37	38	39	40	41
42	43	44	45	46	47	48	49	50	51
52	53	54	55	56	57	58	59	60	61
62	63	64	65	66	67	68	69	70	71
72	73	74	75	76	77	78	79	80	81
82	83	84	85	86	87	88	89	90	91
92	93	94	95	96	97	98	99	100	

To find the primes less than 100, proceed as follows.

i) Strike out all multiples of 2 (4, 6, 8, etc.)

ii) The list's next number that has not been struck out is a prime number.

iii) Strike out from the list all multiples of the number you identified in step (ii).

iv) Repeat steps (ii) and (iii) until you can no longer strike any more multiples.

v) All unstruck numbers in the list are primes.

❧ ❧ ❧ Answers ☙ ☙ ☙

1. 1, 2, 3, 5, 6, 10, 15, 30

3. 1, 83

5. 1, 7, 13, 91

7. 1, 3, 5, 15, 25, 75

9. 1, 2, 4, 8, 16, 32, 64

11. 1, 2, 7, 14

13. 117

15. 13

17. 105

19. 31

21. 364

23. 550

25. 820

27. 493

29. 3797

31. 9701

33. 7517

35. 7033

37. 9896

39. 8758

41. 2363

43. 1269

45. 328

47. 538

49. 586

51. 658

53. 5418

55. 8506

57. 2594

59. 7850

61. 991

63. 937

65. 928

67. 676

69. prime

71. prime

73. composite

75. composite

77. composite

79. prime

81. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 7$

83. $2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

85. $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

87. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$

89. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

91. $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$

93. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$

95. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$

97. $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

99. 100

101. 0

103. 0

105. 4

107. 64

109. 27

111. $784\,\text{in}^2$

113. $484\,\text{in}^2$

115. $12 = 2^2 \cdot 3$

117. $105 = 3 \cdot 5 \cdot 7$

119. $56 = 2^3 \cdot 7$

121. $72 = 2^3 \cdot 3^2$

123. Unstruck numbers are primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

1.5 Order of Operations

The order in which we evaluate expressions can be ambiguous. Take for example, the expression $4 + 3 \cdot 2$. If we do the addition first, then

$$\begin{aligned} 4 + 3 \cdot 2 &= 7 \cdot 2 \\ &= 14. \end{aligned}$$

On the other hand, if we do the multiplication first, then

$$\begin{aligned} 4 + 3 \cdot 2 &= 4 + 6 \\ &= 10. \end{aligned}$$

So, what are we to do?

Of course, grouping symbols can remove the ambiguity.

Grouping Symbols. Parentheses, brackets, or curly braces can be used to group parts of an expression. Each of the following are equivalent:

$$(4+3)\cdot 2 \qquad \text{or} \qquad [4+3]\cdot 2 \qquad \text{or} \qquad \{4+3\}\cdot 2$$

In each case, the rule is "evaluate the expression inside the grouping symbols first." If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.

Thus, for example,

$$\begin{aligned} (4 + 3) \cdot 2 &= 7 \cdot 2 \\ &= 14. \end{aligned}$$

Note how the expression contained in the parentheses was evaluated first.

Another way to avoid ambiguities in evaluating expressions is to establish an order in which operations should be performed. The following guidelines should always be strictly enforced when evaluating expressions.

Rules Guiding Order of Operations. When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.

2. Evaluate all exponents that appear in the expression.

3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.

4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

You Try It!

EXAMPLE 1. Evaluate $4 + 3 \cdot 2$.

Simplify: $8 + 2 \cdot 5$.

Solution. Because of the established *Rules Guiding Order of Operations*, this expression is no longer ambiguous. There are no grouping symbols or exponents, so we immediately go to rule three, evaluate all multiplications and divisions in the order that they appear, moving left to right. After that we invoke rule four, performing all additions and subtractions in the order that they appear, moving left to right.

$$\begin{aligned} 4 + 3 \cdot 2 &= 4 + 6 \\ &= 10 \end{aligned}$$

Thus, $4 + 3 \cdot 2 = 10$.

Answer: 18

You Try It!

EXAMPLE 2. Evaluate $18 - 2 + 3$.

Simplify: $17 - 8 + 2$.

Solution. Follow the *Rules Guiding Order of Operations*. Addition has no precedence over subtraction, nor does subtraction have precedence over addition. We are to perform additions and subtractions as they occur, moving left to right.

$$\begin{aligned} 18 - 2 + 3 &= 16 + 3 && \text{Subtract: } 18 - 2 = 16. \\ &= 19 && \text{Add: } 16 + 3 = 19. \end{aligned}$$

Thus, $18 - 2 + 3 = 19$.

Answer: 11

You Try It!

EXAMPLE 3. Evaluate $54 \div 9 \cdot 2$.

Simplify: $72 \div 9 \cdot 2$.

Solution. Follow the *Rules Guiding Order of Operations*. Division has no precedence over multiplication, nor does multiplication have precedence over division. We are to perform divisions and multiplications as they occur, moving left to right.

$$\begin{aligned} 54 \div 9 \cdot 2 &= 6 \cdot 2 && \text{Divide: } 54 \div 9 = 6. \\ &= 12 && \text{Multiply: } 6 \cdot 2 = 12. \end{aligned}$$

Answer: 16

Thus, $54 \div 9 \cdot 2 = 12$.

□

You Try It!

Simplify: $14 + 3 \cdot 4^2$.

EXAMPLE 4. Evaluate $2 \cdot 3^2 - 12$.

Solution. Follow the *Rules Guiding Order of Operations*, exponents first, then multiplication, then subtraction.

$$\begin{aligned} 2 \cdot 3^2 - 12 &= 2 \cdot 9 - 12 && \text{Evaluate the exponent: } 3^2 = 9. \\ &= 18 - 12 && \text{Perform the multiplication: } 2 \cdot 9 = 18. \\ &= 6 && \text{Perform the subtraction: } 18 - 12 = 6. \end{aligned}$$

Answer: 62

Thus, $2 \cdot 3^2 - 12 = 6$.

□

You Try It!

Simplify: $3(2 + 3 \cdot 4)^2 - 11$.

EXAMPLE 5. Evaluate $12 + 2(3 + 2 \cdot 5)^2$.

Solution. Follow the *Rules Guiding Order of Operations*, evaluate the expression inside the parentheses first, then exponents, then multiplication, then addition.

$$\begin{aligned} 12 + 2(3 + 2 \cdot 5)^2 &= 12 + 2(3 + 10)^2 && \text{Multiply inside parentheses: } 2 \cdot 5 = 10. \\ &= 12 + 2(13)^2 && \text{Add inside parentheses: } 3 + 10 = 13. \\ &= 12 + 2(169) && \text{Exponents are next: } (13)^2 = 169. \\ &= 12 + 338 && \text{Multiplication is next: } 2(169) = 338. \\ &= 350 && \text{Time to add: } 12 + 338 = 350. \end{aligned}$$

Answer: 577

Thus, $12 + 2(3 + 2 \cdot 5)^2 = 350$.

□

You Try It!

EXAMPLE 6. Evaluate $2\{2+2[2+2]\}$.

Simplify: $2\{3+2[3+2]\}$.

Solution. When grouping symbols are nested, evaluate the expression between the pair of innermost grouping symbols first.

$$\begin{aligned} 2\{2+2[2+2]\} &= 2\{2+2[4]\} && \text{Innermost grouping first: } 2+2=4. \\ &= 2\{2+8\} && \text{Multiply next: } 2[4]=8. \\ &= 2\{10\} && \text{Add inside braces: } 2+8=10. \\ &= 20 && \text{Multiply: } 2\{10\}=20 \end{aligned}$$

Thus, $2\{2+2[2+2]\}=20$.

Answer: 26

□

Fraction Bars

Consider the expression

$$\frac{6^2+8^2}{(2+3)^2}.$$

Because a fraction bar means division, the above expression is equivalent to

$$(6^2+8^2) \div (2+3)^2.$$

The position of the grouping symbols signals how we should proceed. We should simplify the numerator, then the denominator, then divide.

Fractional Expressions. If a fractional expression is present, evaluate the numerator and denominator first, then divide.

You Try It!

EXAMPLE 7. Evaluate the expression

$$\frac{6^2+8^2}{(2+3)^2}.$$

Simplify: $\dfrac{12+3\cdot 2}{6}$.

Solution. Simplify the numerator and denominator first, then divide.

$$\begin{aligned} \frac{6^2+8^2}{(2+3)^2} &= \frac{6^2+8^2}{(5)^2} && \text{Parentheses in denominator first: } 2+3=5. \\ &= \frac{36+64}{25} && \text{Exponents are next: } 6^2=36,\ 8^2=64,\ 5^2=25. \\ &= \frac{100}{25} && \text{Add in numerator: } 36+64=100. \\ &= 4 && \text{Divide: } 100\div 25=4. \end{aligned}$$

Answer: 3

Thus, $\dfrac{6^2 + 8^2}{(2+3)^2} = 4.$

The Distributive Property

Consider the expression $2 \cdot (3 + 4)$. If we follow the "Rules Guiding Order of Operations," we would evaluate the expression inside the parentheses first.

$$\begin{aligned} 2 \cdot (3+4) &= 2 \cdot 7 && \text{Parentheses first: } 3 + 4 = 7. \\ &= 14 && \text{Multiply: } 2 \cdot 7 = 14. \end{aligned}$$

However, we could also choose to "distribute" the 2, first multiplying 2 times each addend in the parentheses.

$$\begin{aligned} 2 \cdot (3+4) &= 2 \cdot 3 + 2 \cdot 4 && \text{Multiply 2 times both 3 and 4.} \\ &= 6 + 8 && \text{Multiply: } 2 \cdot 3 = 6 \text{ and } 2 \cdot 4 = 8. \\ &= 14 && \text{Add: } 6 + 8 = 14. \end{aligned}$$

The fact that we get the same answer in the second approach is an illustration of an important property of whole numbers.[1]

The Distributive Property. Let a, b, and c be any whole numbers. Then,

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

We say that "multiplication is distributive with respect to addition."

Multiplication is distributive with respect to addition. If you are not computing the product of a number and a sum of numbers, the distributive property does not apply.

Caution! Wrong Answer Ahead! If you are calculating the product of a number and the product of two numbers, the distributive property must not be used. For example, here is a *common misapplication* of the distributive property.

$$\begin{aligned} 2 \cdot (3 \cdot 4) &= (2 \cdot 3) \cdot (2 \cdot 4) \\ &= 6 \cdot 8 \\ &= 48 \end{aligned}$$

[1] Later, we'll see that this property applies to all numbers, not just whole numbers.

This result is quite distant from the correct answer, which is found by computing the product within the parentheses first.

$$\begin{aligned} 2 \cdot (3 \cdot 4) &= 2 \cdot 12 \\ &= 24. \end{aligned}$$

In order to apply the distributive property, you must be multiplying times a sum.

You Try It!

EXAMPLE 8. Use the distributive property to calculate $4 \cdot (5 + 11)$.

Solution. This is the product of a number and a sum, so the distributive property may be applied.

$$\begin{aligned} 4 \cdot (5 + 11) &= 4 \cdot 5 + 4 \cdot 11 && \text{Distribute the 4 times each addend in the sum.} \\ &= 20 + 44 && \text{Multiply: } 4 \cdot 5 = 20 \text{ and } 4 \cdot 11 = 44. \\ &= 64 && \text{Add: } 20 + 44 = 64. \end{aligned}$$

Readers should check that the same answer is found by computing the sum within the parentheses first.

Distribute: $5 \cdot (11 + 8)$.

Answer: 95

The distributive property is the underpinning of the multiplication algorithm learned in our childhood years.

You Try It!

EXAMPLE 9. Multiply: $6 \cdot 43$.

Solution. We'll express 43 as sum, then use the distributive property.

$$\begin{aligned} 6 \cdot 43 &= 6 \cdot (40 + 3) && \text{Express 43 as a sum: } 43 = 40 + 3. \\ &= 6 \cdot 40 + 6 \cdot 3 && \text{Distribute the 6.} \\ &= 240 + 18 && \text{Multiply: } 6 \cdot 40 = 240 \text{ and } 6 \cdot 3 = 18. \\ &= 258 && \text{Add: } 240 + 18 = 258. \end{aligned}$$

Readers should be able to see this application of the distributive property in the more familiar algorithmic form:

$$\begin{array}{r} 43 \\ \times 6 \\ \hline 18 \\ 240 \\ \hline 258 \end{array}$$

Use the distributive property to evaluate $8 \cdot 92$.

Or in the even more condensed form with "carrying:"

$$\begin{array}{r} {}^{1}43 \\ \times 6 \\ \hline 2\,5\,8 \end{array}$$

Answer: 736

Multiplication is also distributive with respect to subtraction.

The Distributive Property (Subtraction). Let a, b, and c be any whole numbers. Then,

$$a \cdot (b - c) = a \cdot b - a \cdot c.$$

We say the multiplication is "distributive with respect to subtraction."

You Try It!

Distribute: $8 \cdot (9 - 2)$.

EXAMPLE 10. Use the distributive property to simplify: $3 \cdot (12 - 8)$.

Solution. This is the product of a number and a difference, so the distributive property may be applied.

$$\begin{aligned} 3 \cdot (12 - 8) &= 3 \cdot 12 - 3 \cdot 8 && \text{Distribute the 3 times each term in the difference.} \\ &= 36 - 24 && \text{Multiply: } 3 \cdot 12 = 36 \text{ and } 3 \cdot 8 = 24. \\ &= 12 && \text{Subtract: } 36 - 24 = 12. \end{aligned}$$

Alternate solution. Note what happens if we use the usual "order of operations" to evaluate the expression.

$$\begin{aligned} 3 \cdot (12 - 8) &= 3 \cdot 4 && \text{Parentheses first: } 12 - 8 = 4. \\ &= 12 && \text{Multiply: } 3 \cdot 4 = 12. \end{aligned}$$

Same answer.

Answer: 56

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-12, simplify the given expression.

1. $5 + 2 \cdot 2$

2. $5 + 2 \cdot 8$

3. $23 - 7 \cdot 2$

4. $37 - 3 \cdot 7$

5. $4 \cdot 3 + 2 \cdot 5$

6. $2 \cdot 5 + 9 \cdot 7$

7. $6 \cdot 5 + 4 \cdot 3$

8. $5 \cdot 2 + 9 \cdot 8$

9. $9 + 2 \cdot 3$

10. $3 + 6 \cdot 6$

11. $32 - 8 \cdot 2$

12. $24 - 2 \cdot 5$

In Exercises 13-28, simplify the given expression.

13. $45 \div 3 \cdot 5$

14. $20 \div 1 \cdot 4$

15. $2 \cdot 9 \div 3 \cdot 18$

16. $19 \cdot 20 \div 4 \cdot 16$

17. $30 \div 2 \cdot 3$

18. $27 \div 3 \cdot 3$

19. $8 - 6 + 1$

20. $15 - 5 + 10$

21. $14 \cdot 16 \div 16 \cdot 19$

22. $20 \cdot 17 \div 17 \cdot 14$

23. $15 \cdot 17 + 10 \div 10 - 12 \cdot 4$

24. $14 \cdot 18 + 9 \div 3 - 7 \cdot 13$

25. $22 - 10 + 7$

26. $29 - 11 + 1$

27. $20 \cdot 10 + 15 \div 5 - 7 \cdot 6$

28. $18 \cdot 19 + 18 \div 18 - 6 \cdot 7$

In Exercises 29-40, simplify the given expression.

29. $9 + 8 \div \{4 + 4\}$

30. $10 + 20 \div \{2 + 2\}$

31. $7 \cdot [8 - 5] - 10$

32. $11 \cdot [12 - 4] - 10$

33. $(18 + 10) \div (2 + 2)$

34. $(14 + 7) \div (2 + 5)$

35. $9 \cdot (10 + 7) - 3 \cdot (4 + 10)$

36. $9 \cdot (7 + 7) - 8 \cdot (3 + 8)$

37. $2 \cdot \{8 + 12\} \div 4$

38. $4 \cdot \{8 + 7\} \div 3$

39. $9 + 6 \cdot (12 + 3)$

40. $3 + 5 \cdot (10 + 12)$

In Exercises 41-56, simplify the given expression.

41. $2 + 9 \cdot [7 + 3 \cdot (9 + 5)]$

42. $6 + 3 \cdot [4 + 4 \cdot (5 + 8)]$

43. $7 + 3 \cdot [8 + 8 \cdot (5 + 9)]$

44. $4 + 9 \cdot [7 + 6 \cdot (3 + 3)]$

45. $6 - 5[11 - (2 + 8)]$

46. $15 - 1[19 - (7 + 3)]$

47. $11 - 1[19 - (2 + 15)]$

48. $9 - 8[6 - (2 + 3)]$

49. $4\{7[9 + 3] - 2[3 + 2]\}$

50. $4\{8[3 + 9] - 4[6 + 2]\}$

51. $9 \cdot [3 + 4 \cdot (5 + 2)]$

52. $3 \cdot [4 + 9 \cdot (8 + 5)]$

53. $3\{8[6 + 5] - 8[7 + 3]\}$

54. $2\{4[6 + 9] - 2[3 + 4]\}$

55. $3 \cdot [2 + 4 \cdot (9 + 6)]$

56. $8 \cdot [3 + 9 \cdot (5 + 2)]$

In Exercises 57-68, simplfiy the given expression.

57. $(5 - 2)^2$

58. $(5 - 3)^4$

59. $(4 + 2)^2$

60. $(3 + 5)^2$

61. $2^3 + 3^3$

62. $5^4 + 2^4$

63. $2^3 - 1^3$

64. $3^2 - 1^2$

65. $12 \cdot 5^2 + 8 \cdot 9 + 4$

66. $6 \cdot 3^2 + 7 \cdot 5 + 12$

67. $9 - 3 \cdot 2 + 12 \cdot 10^2$

68. $11 - 2 \cdot 3 + 12 \cdot 4^2$

In Exercises 69-80, simplify the given expression.

69. $4^2 - (13 + 2)$

70. $3^3 - (7 + 6)$

71. $3^3 - (7 + 12)$

72. $4^3 - (6 + 5)$

73. $19 + 3[12 - (2^3 + 1)]$

74. $13 + 12[14 - (2^2 + 1)]$

75. $17 + 7[13 - (2^2 + 6)]$

76. $10 + 1[16 - (2^2 + 9)]$

77. $4^3 - (12 + 1)$

78. $5^3 - (17 + 15)$

79. $5 + 7[11 - (2^2 + 1)]$

80. $10 + 11[20 - (2^2 + 1)]$

In Exercises 81-92, simplify the given expression.

81. $\dfrac{13+35}{3(4)}$

82. $\dfrac{35+28}{7(3)}$

83. $\dfrac{64-(8\cdot 6-3)}{4\cdot 7-9}$

84. $\dfrac{19-(4\cdot 3-2)}{6\cdot 3-9}$

85. $\dfrac{2+13}{4-1}$

86. $\dfrac{7+1}{8-4}$

87. $\dfrac{17+14}{9-8}$

88. $\dfrac{16+2}{13-11}$

89. $\dfrac{37+27}{8(2)}$

90. $\dfrac{16+38}{6(3)}$

91. $\dfrac{40-(3\cdot 7-9)}{8\cdot 2-2}$

92. $\dfrac{60-(8\cdot 6-3)}{5\cdot 4-5}$

In Exercises 93-100, use the distributive property to evaluate the given expression.

93. $5\cdot(8+4)$

94. $8\cdot(4+2)$

95. $7\cdot(8-3)$

96. $8\cdot(9-7)$

97. $6\cdot(7-2)$

98. $4\cdot(8-6)$

99. $4\cdot(3+2)$

100. $4\cdot(9+6)$

In Exercises 101-104, use the distributive property to evaluate the given expression using the technique shown in Example 9.

101. $9\cdot 62$

102. $3\cdot 76$

103. $3\cdot 58$

104. $7\cdot 57$

❧ ❧ ❧ Answers ☙ ☙ ☙

1. 9

3. 9

5. 22

7. 42

9. 15

11. 16

13. 75

15. 108

17. 45

19. 3

21. 266

23. 208

25. 19

27. 161

29. 10

31. 11

33. 7

35. 111

37. 10

39. 99

41. 443

43. 367

45. 1

47. 9

49. 296

51. 279

53. 24

55. 186

57. 9

59. 36

61. 35

63. 7

65. 376

67. 1203

69. 1

71. 8

73. 28

75. 38

77. 51

79. 47

81. 4

83. 1

85. 5

87. 31

89. 4

91. 2

93. 60

95. 35

97. 30

99. 20

101. 558

103. 174

1.6 Solving Equations by Addition and Subtraction

Let's start with the definition of a variable.

Variable. A *variable* is a symbol (usually a letter) that stands for a value that may vary.

Next we follow with the definition of an equation.

Equation. An *equation* is a mathematical statement that equates two mathematical expressions.

The key difference between a mathematical expression and an equation is the presence of an an equals sign. So, for example,

$$2+3[5-4\cdot 2], \qquad x^2+2x-3, \qquad \text{and} \qquad x+2y+3$$

are mathematical expressions (two of which contain variables), while

$$3+2(7-3)=11, \qquad x+3=4, \qquad \text{and} \qquad 3x=9$$

are equations. Note that each of the equations contain an equals sign, but the expressions do not.

Next we have the definition of a solution of an equation.

What it Means to be a Solution. A *solution* of an equation is a numerical value that satisfies the equation. That is, when the variable in the equation is replaced by the solution, a true statement results.

You Try It!

Show that 27 is a solution of the equation $x-12=15$.

EXAMPLE 1. Show that 3 is a solution of the equation $x+8=11$.

Solution. Substitute 3 for x in the given equation and simplify.

$x+8=11$	The given equation.
$3+8=11$	Substitute 3 for x.
$11=11$	Simplify both sides.

Since the left- and right-hand sides of the last line are equal, this shows that when 3 is substituted for x in the equation a true statement results. Therefore, 3 is a solution of the equation.

□

You Try It!

Is 8 a solution of $5 = 12 - y$?

EXAMPLE 2. Is 23 a solution of the equation $4 = y - 11$?

Solution. Substitute 23 for y in the given equation and simplify.

$4 = y - 11$	The given equation.
$4 = 23 - 11$	Substitute 23 for y.
$4 = 12$	Simplify both sides.

Since the left- and right-hand sides of the last line are **not** equal, this shows that when 23 is substituted for y in the equation a false statement results. Therefore, 23 is **not** a solution of the equation.

Answer: No.

□

Equivalent Equations

We start with the definition of equivalent equations.

Equivalent Equations. Two equations are equivalent if they have the same solution set.

You Try It!

Are the equations $x = 4$ and $x + 8 = 3$ equivalent?

EXAMPLE 3. Are the equations $x + 2 = 9$ and $x = 7$ equivalent?

Solution. The number 7 is the only solution of the equation $x + 2 = 9$. Similarly, 7 is the only solution of the equation $x = 7$. Therefore $x + 2 = 9$ and $x = 7$ have the same solution sets and are equivalent.

Answer: No.

□

You Try It!

Are the equations $x = 2$ and $x^2 = 2x$ equivalent?

EXAMPLE 4. Are the equations $x^2 = x$ and $x = 1$ equivalent?

Solution. By inspection, the equation $x^2 = x$ has two solutions, 0 and 1. On the other hand, the equation $x = 1$ has a single solution, namely 1. Hence, the equations $x^2 = x$ and $x = 1$ do not have the same solution sets and are **not** equivalent.

Answer: No.

□

Operations that Produce Equivalent Equations

There are many operations that will produce equivalent operations. In this section we look at two: addition and subtraction.

Adding the Same Quantity to Both Sides of an Equation. Adding the same quantity to both sides of an equation does not change the solution set. That is, if

$$a = b,$$

then adding c to both sides of the equation produces the equivalent equation

$$a + c = b + c.$$

Let's see if this works as advertised. Consider the equaton

$$x - 4 = 3.$$

By inspection, 7 is the only solution of the equation. Now, let's add 4 to both sides of the equation to see if the resulting equation is equivalent to $x - 4 = 3$.

$x - 4 = 3$	The given equation.
$x - 4 + 4 = 3 + 4$	Add 4 to both sides of the equation.
$x = 7$	Simplify both sides of the equation.

The number 7 is the only solution of the equation $x = 7$. Thus, the equation $x = 7$ is equivalent to the original equation $x - 4 = 3$ (they have the same solutions).

Important Point. Adding the same amount to both sides of an equation does not change its solutions.

It is also a fact that subtracting the same quantity from both sides of an equation produces an equivalent equation.

Subtracting the Same Quantity from Both Sides of an Equation. Subtracting the same quantity from both sides of an equation does not change the solution set. That is, if

$$a = b,$$

then subtracting c from both sides of the equation produces the equivalent equation

$$a - c = b - c.$$

Let's also see if this works as advertised. Consider the equation

$$x + 4 = 9.$$

By inspection, 5 is the only solution of the equation. Now, let's subtract 4 from both sides of the equation to see if the resulting equation is equivalent to $x + 4 = 9$.

$x + 4 = 9$	The given equation.
$x + 4 - 4 = 9 - 4$	Subtract 4 from both sides of the equation.
$x = 5$	Simplify both sides of the equation.

The number 5 is the only solution of the equation $x = 5$. Thus, the equation $x = 5$ is equivalent to the original equation $x + 4 = 9$ (they have the same solutions).

Important Point. Subtracting the same amount from both sides of an equation does not change its solutions.

Writing Mathematics. When solving equations, observe the following rules to neatly arrange your work:

1. **One equation per line.** This means that you should not arrange your work like this:

   $$x + 3 = 7 \qquad x + 3 - 3 = 7 - 3 \qquad x = 4$$

   That's three equations on a line. Rather, arrange your work one equation per line like this:

   $$\begin{aligned} x + 3 &= 7 \\ x + 3 - 3 &= 7 - 3 \\ x &= 4 \end{aligned}$$

2. **Add and subtract inline.** Don't do this:

   $$\begin{array}{r} x - 7 = \ 12 \\ +7 \quad +7 \\ \hline x = \ 19 \end{array}$$

   Instead, add 7 to both sides of the equation "inline."

   $$\begin{aligned} x - 7 &= 12 \\ x - 7 + 7 &= 12 + 7 \\ x &= 19 \end{aligned}$$

Wrap and Unwrap

Suppose that you are wrapping a gift for your cousin. You perform the following steps in order.

1. Put the gift paper on.
2. Put the tape on.
3. Put the decorative bow on.

When we give the wrapped gift to our cousin, he politely unwraps the present, "undoing" each of our three steps in inverse order.

1. Take off the decorative bow.
2. Take off the tape.
3. Take off the gift paper.

This seemingly frivolous wrapping and unwrapping of a gift contains some deeply powerful mathematical ideas.

Consider the mathematical expression $x+4$. To evaluate this expression at a particular value of x, we would start with the given value of x, then

1. Add 4.

Suppose we started with the number 7. If we add 4, we arrive at the following result: 11.

Now, how would we "unwrap" this result to return to our original number? We would start with our result, then

1. Subtract 4.

That is, we would take our result from above, 11, then subtract 4, which returns us to our original number, namely 7.

Addition and Subtraction as Inverse Operations. Two extremely important observations:

The inverse of addition is subtraction. If we start with a number x and add a number a, then subtracting a from the result will return us to the original number x. In symbols,

$$x + a - a = x.$$

The inverse of subtraction is addition. If we start with a number x and subtract a number a, then adding a to the result will return us to the original number x. In symbols,

$$x - a + a = x.$$

You Try It!

Solve $x + 5 = 12$ for x.

EXAMPLE 5. Solve $x - 8 = 10$ for x.

Solution. To undo the effects of subtracting 8, we add 8 to both sides of the equation.

$x - 8 = 10$	Original equation.
$x - 8 + 8 = 10 + 8$	Add 8 to both sides of the equation.
$x = 18$	On the left, adding 8 "undoes" the effect of subtracting 8 and returns x. On the right, 10+8=18.

Therefore, the solution of the equation is 18.

Check. To check, substitute the solution 18 into the original equation.

$x - 8 = 10$	Original equation.
$18 - 8 = 10$	Substitute 18 for x.
$10 = 10$	Simplify both sides.

The fact that the last line of our check is a true statement guarantees that 18 is a solution of $x - 8 = 10$.

Answer: $x = 7$.

□

You Try It!

Solve $y - 8 = 11$ for y.

EXAMPLE 6. Solve $11 = y + 5$ for y.

Solution. To undo the effects of adding 5, we subtract 5 from both sides of the equation.

$11 = y + 5$	Original equation.
$11 - 5 = y + 5 - 5$	Subtract 5 from both sides of the equation.
$6 = y$	On the right, subtracting "undoes" the effect of adding 5 and returns y. On the left, 11-5=6.

Therefore, the solution of the equation is 6.

Check. To check, substitute the solution 6 into the original equation.

$11 = y + 5$	Original equation.
$11 = 6 + 5$	Substitute 6 for y.
$11 = 11$	Simplify both sides.

The fact that the last line of our check is a true statement guarantees that 6 is a solution of $11 = y + 5$.

Answer: $y = 19$.

□

Word Problems

The solution of a word problem must incorporate each of the following steps.

Requirements for Word Problem Solutions.

1. **Set up a Variable Dictionary.** You must let your readers know what each variable in your problem represents. This can be accomplished in a number of ways:
   - Statements such as "Let P represent the perimeter of the rectangle."
   - Labeling unknown values with variables in a table.
   - Labeling unknown quantities in a sketch or diagram.
2. **Set up an Equation.** Every solution to a word problem must include a carefully crafted equation that accurately describes the constraints in the problem statement.
3. **Solve the Equation.** You must always solve the equation set up in the previous step.
4. **Answer the Question.** This step is easily overlooked. For example, the problem might ask for Jane's age, but your equation's solution gives the age of Jane's sister Liz. Make sure you answer the original question asked in the problem. Your solution should be written in a sentence with appropriate units.
5. **Look Back.** It is important to note that this step does not imply that you should simply check your solution in your equation. After all, it's possible that your equation incorrectly models the problem's situation, so you could have a valid solution to an incorrect equation. The important question is: "Does your answer make sense based on the words in the original problem statement."

Let's give these requirements a test drive.

You Try It!

12 more than a certain number is 19. Find the number.

EXAMPLE 7. Four more than a certain number is 12. Find the number.

Solution. In our solution, we will carefully address each step of the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* We can satisfy this requirement by simply stating "Let x represent a certain number."

2. *Set up an Equation.* "Four more than a certain number is 12" becomes

4	more than	a certain number	is	12
4	$+$	x	$=$	12

3. *Solve the Equation.* To "undo" the addition, subtract 4 from both sides of the equation.

$$\begin{aligned} 4 + x &= 12 && \text{Original equation.} \\ 4 + x - 4 &= 12 - 4 && \text{Subtract 4 from both sides of the equation.} \\ x &= 8 && \text{On the left, subtracting 4 "undoes" the effect of adding 4 and returns } x\text{. On the right, } 12 - 4 = 8. \end{aligned}$$

4. *Answer the Question.* The number is 8.

5. *Look Back.* Does the solution 8 satisfy the words in the original problem? We were told that "four more than a certain number is 12." Well, four more than 8 is 12, so our solution is correct.

Answer: 7

□

You Try It!

Fred withdraws $230 from his account, lowering his balance to $3,500. What was his original balance?

EXAMPLE 8. Amelie withdraws $125 from her savings account. Because of the withdrawal, the current balance in her account is now $1,200. What was the original balance in the account before the withdrawal?

Solution. In our solution, we will carefully address each step of the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* We can satisfy this requirement by simply stating "Let B represent the original balance in Amelie's account."

2. *Set up an Equation.* We can describe the situation in words and symbols.

Original Balance	minus	Amelie's Withdrawal	is	Current Balance
B	$-$	125	$=$	1200

3. *Solve the Equation.* To "undo" the subtraction, add 125 to both sides of the equation.

$B - 125 = 1200$	Original equation.
$B - 125 + 125 = 1200 + 125$	Add 125 to both sides of the equation.
$B = 1325$	On the left, adding 125 "undoes" the effect of subtracting 125 and returns B. On the right, 1200+125=1325.

4. *Answer the Question.* The original balance was $1,325.

5. *Look Back.* Does the solution $1,325 satisfy the words in the original problem? Note that if Amelie withdraws $125 from this balance, the new balance will be $1,200. Hence, the solution is correct.

Answer: $3,730.

□

You Try It!

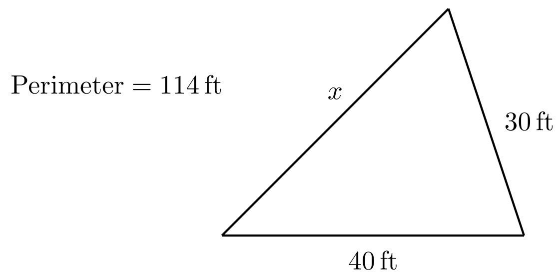
The perimeter of a quadrilater is 200 meters. If three of the sides measure 20, 40, and 60 meters, what is the length of the fourth side?

EXAMPLE 9. The perimeter of a triangle is 114 feet. Two of the sides of the triangle measure 30 feet and 40 feet, respectively. Find the measure of the third side of the triangle.

Movie

Solution. In our solution, we will carefully address each step of the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* When geometry is involved, we can create our variable dictionary by labeling a carefully constructed diagram. With this thought in mind, we draw a triangle, then label its known and unknown sides and its perimeter.



The figure makes it clear that x represents the length of the unknown side of the triangle. The figure also summarizes information needed for the solution.

2. *Set up an equation.* We know that the perimeter of a triangle is found by finding the sum of its three sides; in words and symbols,

Perimeter	is	First Side	plus	Second Side	plus	Third Side
114	=	x	+	30	+	40

Simplify the right-hand side by adding 30 and 40; i.e., $30 + 40 = 70$.

$$114 = x + 70$$

3. *Solve the Equation.* To "undo" adding 70, subtract 70 from both sides of the equation.

$114 = x + 70$	Our equation.
$114 - 70 = x + 70 - 70$	Subtract 70 from both sides.
$44 = x$	On the right, subtracting 70 "undoes" the effect of adding 70 and returns to x. On the left, $114 - 70 = 44$.

4. *Answer the Question.* The unknown side of the triangle is 44 feet.

5. *Look Back.* Does the solution 44 feet satisfy the words of the original problem? We were told that the perimeter is 114 feet and two of the sides have length 30 feet and 40 feet respectively. We found that the third side has length 44 feet. Now, adding the three sides, $30+40+44 = 114$, which equals the given perimeter of 114 feet. The answer works!

Answer: 80 meters.

□

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-12, which of the numbers following the given equation are solutions of the given equation? Support your response with work similar to that shown in Examples 1 and 2.

1. $x - 4 = 6$; 10, 17, 13, 11

2. $x - 9 = 7$; 17, 23, 19, 16

3. $x + 2 = 6$; 5, 11, 7, 4

4. $x + 3 = 9$; 6, 9, 7, 13

5. $x + 2 = 3$; 8, 1, 4, 2

6. $x + 2 = 5$; 10, 3, 6, 4

7. $x - 4 = 7$; 12, 11, 18, 14

8. $x - 6 = 7$; 13, 16, 20, 14

9. $x + 3 = 4$; 8, 4, 2, 1

10. $x + 5 = 9$; 5, 11, 7, 4

11. $x - 6 = 8$; 17, 21, 14, 15

12. $x - 2 = 9$; 11, 14, 12, 18

In Exercises 13-52, solve the given equation for x.

13. $x + 5 = 6$

14. $x + 6 = 19$

15. $5 = 4 + x$

16. $10 = 8 + x$

17. $13 + x = 17$

18. $7 + x = 15$

19. $9 + x = 10$

20. $14 + x = 17$

21. $19 = x - 3$

22. $2 = x - 11$

23. $x - 18 = 1$

24. $x - 20 = 8$

25. $x - 3 = 11$

26. $x - 17 = 18$

27. $2 + x = 4$

28. $1 + x = 16$

29. $x - 14 = 12$

30. $x - 1 = 17$

31. $x + 2 = 8$

32. $x + 11 = 14$

33. $11 + x = 17$

34. $11 + x = 18$

35. $x + 13 = 17$

36. $x + 1 = 16$

37. $20 = 3 + x$

38. $9 = 3 + x$

39. $20 = 8 + x$

40. $10 = 3 + x$

41. $3 = x - 20$

42. $13 = x - 15$

43. $x + 16 = 17$

44. $x + 6 = 12$

45. $5 = x - 6$

46. $10 = x - 7$

47. $18 = x - 6$

48. $14 = x - 4$

49. $18 = 13 + x$

50. $17 = 5 + x$

51. $x - 9 = 15$

52. $x - 11 = 17$

53. 12 less than a certain number is 19. Find the number.

54. 19 less than a certain number is 1. Find the number.

55. A triangle has a perimeter of 65 feet. It also has two sides measuring 19 feet and 17 feet, respectively. Find the length of the third side of the triangle.

56. A triangle has a perimeter of 55 feet. It also has two sides measuring 14 feet and 13 feet, respectively. Find the length of the third side of the triangle.

57. Burt makes a deposit to an account having a balance of $1900. After the deposit, the new balance in the account is $8050. Find the amount of the deposit.

58. Dave makes a deposit to an account having a balance of $3500. After the deposit, the new balance in the account is $4600. Find the amount of the deposit.

59. 8 more than a certain number is 18. Find the number.

60. 3 more than a certain number is 19. Find the number.

61. Michelle withdraws a $120 from her bank account. As a result, the new account balance is $1000. Find the account balance before the withdrawal.

62. Mercy withdraws a $430 from her bank account. As a result, the new account balance is $1200. Find the account balance before the withdrawal.

63. **Foreclosures.** Between January and March last year, 650,000 homes received a foreclosure notice. Between the first three months of this year, there were 804,000 foreclosure notices. What was the increase in home foreclosure notices? *Associated Press Times-Standard 4/22/09*

64. **Home Price.** According to the Humboldt State University Economics Department's Humboldt Economic Index, the median home price in the US fell $1500 over the last month to $265,000. What was the median home price before the price drop?

65. **Unmanned Areal Vehicle.** Northrup Grumman's Global Hawk unmanned drone can fly at 65,000 feet, 40,000 feet higher than NASA's Ikhana unmanned aircraft. How high can the Ikhana fly?

66. **Tribal Land.** The Yurok Tribe has the option to purchase 47,000 acres in order to increase its ancestral territory. The first phase would include 22,500 acres in the Cappel and Pecman watersheds. The second phase plans for acreage in the Blue Creek area. How many acres could be purchased in the second phase? *Times-Standard 4/15/09*

❧ ❧ ❧ Answers ☙ ☙ ☙

1. 10

3. 4

5. 1

7. 11

9. 1

11. 14

13. 1

15. 1

17. 4

19. 1

21. 22

23. 19

25. 14

27. 2

29. 26

31. 6

33. 6

35. 4

37. 17

39. 12

41. 23

43. 1

45. 11

47. 24

49. 5

51. 24

53. 31

55. 29

57. $6150

59. 10

61. $1120

63. 154,000

65. 25,000 feet

1.7 Solving Equations by Multiplication and Division

In Section 1.6, we stated that two equations that have the same solutions are *equivalent*. Furthermore, we saw that adding the same number to both sides of an equation produced an equivalent equation. Similarly, subtracting the same the number from both sides of an equation also produces an equivalent equation. We can make similar statements for multiplication and division.

Multiplying both Sides of an Equation by the Same Quantity. Multiplying both sides of an equation by the same quantity does not change the solution set. That is, if

$$a = b,$$

then multiplying both sides of the equation by c produces the equivalent equation

$$a \cdot c = b \cdot c,$$

provided $c \neq 0$.

A similar statement can be made about division.

Dividing both Sides of an Equation by the Same Quantity. Dividing both sides of an equation by the same quantity does not change the solution set. That is, if

$$a = b,$$

then dividing both sides of the equation by c produces the equivalent equation

$$\frac{a}{c} = \frac{b}{c},$$

provided $c \neq 0$.

In Section 1.6, we saw that addition and subtraction were inverse operations. If you start with a number, add 4 and subtract 4, you are back to the original number. This concept also works for multiplication and division.

Multiplication and Division as Inverse Operations. Two extremely important observations:

The inverse of multiplication is division. If we start with a number x and multiply by a number a, then dividing the result by the number a returns us to the original number x. In symbols,

$$\frac{a \cdot x}{a} = x.$$

The inverse of division is multiplication. If we start with a number x and divide by a number a, then multiplying the result by the number a returns us to the original number x. In symbols,

$$a \cdot \frac{x}{a} = x.$$

Let's put these ideas to work.

You Try It!

Solve for x: $5x = 120$

EXAMPLE 1. Solve the equation $3x = 24$ for x.

Solution. To undo the effects of multiplying by 3, we divide both sides of the equation by 3.

$3x = 24$	Original equation.
$\dfrac{3x}{3} = \dfrac{24}{3}$	Divide both sides of the equation by 3.
$x = 8$	On the left, dividing by 3 "undoes" the effect of multiplying by 3 and returns to x. On the right, $24/3 = 8$.

Solution. To check, substitute the solution 8 into the original equation.

$3x = 24$	Original equation.
$3(8) = 24$	Substitute 8 for x.
$24 = 24$	Simplify both sides.

That fact that the last line of our check is a true statement guarantees that 8 is a solution of $3x = 24$.

Answer: 24

□

You Try It!

Solve for x: $x/2 = 19$

EXAMPLE 2. Solve the following equation for x.

$$\frac{x}{7} = 12$$

Solution. To undo the effects of dividing by 7, we multiply both sides of the equation by 7.

$\dfrac{x}{7} = 12$	Original equation.
$7 \cdot \dfrac{x}{7} = 7 \cdot 12$	Multiply both sides of the equation by 7.
$x = 84$	On the left, multiplying by 7 "undoes" the effect of dividing by 7 and returns to x. On the right, $7 \cdot 12 = 84$.

Solution. To check, substitute the solution 84 into the original equation.

$\dfrac{x}{7} = 12$	Original equation.
$\dfrac{84}{7} = 12$	Substitute 84 for x.
$12 = 12$	Simplify both sides.

That fact that the last line of our check is a true statement guarantees that 84 is a solution of $x/7 = 12$.

Answer: 38

Word Problems

In Section 1.6 we introduced *Requirements for Word Problem Solutions.* Those requirements will be strictly adhered to in this section.

You Try It!

Seven times a certain number is one hundred five. Find the unknown number.

EXAMPLE 3. Fifteen times a certain number is 45. Find the unknown number.

Solution. In our solution, we will carefully address each step of the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* We can satisfy this requirement by simply stating "Let x represent a certain number."

2. *Set up an equation.* "Fifteen times a certain number is 45" becomes

15	times	a certain number	is	45
15	$\cdot$	x	$=$	45

3. *Solve the Equation.* To "undo" the multiplication by 15, divide both sides of the equation by 15.

$15x = 45$	Original equation. Write $15 \cdot x$ as $15x$
$\dfrac{15x}{15} = \dfrac{45}{15}$	Divide both sides of the equation by 15.
$x = 3$	On the left, dividing by 15 "undoes" the effect of multiplying by 15 and returns to x. On the right, $45/15 = 3$.

4. *Answer the Question.* The unknown number is 3.

5. *Look Back.* Does the solution 3 satisfy the words of the original problem? We were told that "15 times a certain number is 45." Well, 15 times 3 is 45, so our solution is correct.

Answer: 15

□

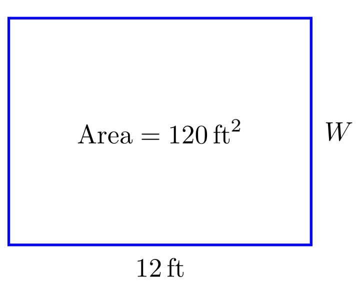
You Try It!

The area of a rectangle is 3,500 square meters. If the width is 50 meters, find the length.

EXAMPLE 4. The area of a rectangle is 120 square feet. If the length of the rectangle is 12 feet, find the width of the rectangle.

Solution. In our solution, we will carefully address each step of the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* When geometry is involved, we can create our variable dictionary by labeling a carefully constructed diagram. With this thought in mind, we draw a rectangle, then label its length, width, and area.



The figure makes it clear that W represents the width of the rectangle. The figure also summarizes information needed for the solution.

2. *Set up an equation.* We know that the area of a rectangle is found by multiplying its length and width; in symbols,

$$A = LW. \tag{1.1}$$

We're given the area is $A = 120\,\text{ft}^2$ and the length is $L = 12\,\text{ft}$. Substitute these numbers into the area formula (1.1) to get

$$120 = 12W.$$

3. *Solve the Equation.* To "undo" the multiplication by 12, divide both sides of the equation by 12.

$120 = 12W$	Our equation.
$\dfrac{120}{12} = \dfrac{12W}{12}$	Divide both sides of the equation by 12.
$10 = W$	On the right, dividing by 12 "undoes" the effect of multiplying by 12 and returns to W. On the left, $120/12 = 10$.

4. *Answer the Question.* The width is 10 feet.

5. *Look Back.* Does the found width satisfy the words of the original problem? We were told that the area is 120 square feet and the length is 12 feet. The area is found by multiplying the length and width, which gives us 12 feet times 10 feet, or 120 square feet. The answer works!

Answer: 70 meters

□

You Try It!

A class of 30 students averaged 75 points on an exam. How many total points were accumulated by the class as a whole?

EXAMPLE 5. A class of 23 students averaged 76 points on an exam.How many total points were accumulated by the class as a whole?

Movie

Solution. In our solution, we will carefully address each step of the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* We can set up our variable dictionary by simply stating "Let T represent the total points accumulated by the class."

2. *Set up an equation.* To find the average score on the exam, take the total points accumulated by the class, then divide by the number of students in the class. In words and symbols,

Total Points	divided by	Number of Students	equals	Average Score
T	$\div$	23	$=$	76

An equivalent representation is

$$\frac{T}{23} = 76.$$

3. *Solve the Equation.* To "undo" the division by 23, multiply both sides of the equation by 23.

$$\frac{T}{23} = 76 \qquad \text{Our equation.}$$

$$23 \cdot \frac{T}{23} = 76 \cdot 23 \qquad \text{Multiply both sides of the equation by 23.}$$

$$T = 1748$$ On the left, multiplying by 23 "undoes" the effect of dividing by 23 and returns to T. On the right, $76 \cdot 23 = 1748$.

4. *Answer the Question.* The total points accumulated by the class on the exam is 1,748.

5. *Look Back.* Does the solution 1,748 satisfy the words of the original problem? To find the average on the exam, divide the total points 1,748 by 23, the number of students in the class. Note that this gives an average score of $1748 \div 23 = 76$. The answer works!

Answer: 2,250

□

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-12, which of the numbers following the given equation are solutions of the given equation?

1. $\frac{x}{6} = 4$; 24, 25, 27, 31

2. $\frac{x}{7} = 6$; 49, 42, 43, 45

3. $\frac{x}{2} = 3$; 6, 9, 13, 7

4. $\frac{x}{9} = 5$; 45, 46, 48, 52

5. $5x = 10$; 9, 2, 3, 5

6. $4x = 36$; 12, 16, 9, 10

7. $5x = 25$; 5, 6, 8, 12

8. $3x = 3$; 1, 8, 4, 2

9. $2x = 2$; 4, 8, 1, 2

10. $3x = 6$; 2, 9, 5, 3

11. $\frac{x}{8} = 7$; 57, 59, 63, 56

12. $\frac{x}{3} = 7$; 24, 21, 28, 22

In Exercises 13-36, solve the given equation for x.

13. $\frac{x}{6} = 7$

14. $\frac{x}{8} = 6$

15. $2x = 16$

16. $2x = 10$

17. $2x = 18$

18. $2x = 0$

19. $4x = 24$

20. $2x = 4$

21. $\frac{x}{4} = 9$

22. $\frac{x}{5} = 6$

23. $5x = 5$

24. $3x = 15$

25. $5x = 30$

26. $4x = 28$

27. $\frac{x}{3} = 4$

28. $\frac{x}{9} = 4$

29. $\frac{x}{8} = 9$

30. $\frac{x}{8} = 2$

31. $\frac{x}{7} = 8$

32. $\frac{x}{4} = 6$

33. $2x = 8$

34. $3x = 9$

35. $\frac{x}{8} = 5$

36. $\frac{x}{5} = 4$

37. The price of one bookcase is $370. A charitable organization purchases an unknown number of bookcases and the total price of the purchase is $4,810. Find the number of bookcases purchased.

38. The price of one computer is $330. A charitable organization purchases an unknown number of computers and the total price of the purchase is $3,300. Find the number of computers purchased.

39. When an unknown number is divided by 3, the result is 2. Find the unknown number.

40. When an unknown number is divided by 8, the result is 3. Find the unknown number.

41. A class of 29 students averaged 80 points on an exam. How many total points were accumulated by the class as a whole?

42. A class of 44 students averaged 87 points on an exam. How many total points were accumulated by the class as a whole?

43. When an unknown number is divided by 9, the result is 5. Find the unknown number.

44. When an unknown number is divided by 9, the result is 2. Find the unknown number.

45. The area of a rectangle is 16 square cm. If the length of the rectangle is 2 cm, find the width of the rectangle.

46. The area of a rectangle is 77 square ft. If the length of the rectangle is 7 ft, find the width of the rectangle.

47. The area of a rectangle is 56 square cm. If the length of the rectangle is 8 cm, find the width of the rectangle.

48. The area of a rectangle is 55 square cm. If the length of the rectangle is 5 cm, find the width of the rectangle.

49. The price of one stereo is $430. A charitable organization purchases an unknown number of stereos and the total price of the purchase is $6,020. Find the number of stereos purchased.

50. The price of one computer is $490. A charitable organization purchases an unknown number of computers and the total price of the purchase is $5,880. Find the number of computers purchased.

51. A class of 35 students averaged 74 points on an exam. How many total points were accumulated by the class as a whole?

52. A class of 44 students averaged 88 points on an exam. How many total points were accumulated by the class as a whole?

53. 5 times an unknown number is 20. Find the unknown number.

54. 5 times an unknown number is 35. Find the unknown number.

55. 3 times an unknown number is 21. Find the unknown number.

56. 2 times an unknown number is 10. Find the unknown number.

❧ ❧ ❧ Answers ☙ ☙ ☙

1. 24

3. 6

5. 2

7. 5

9. 1

11. 56

13. 42

15. 8

17. 9

19. 6

21. 36

23. 1

25. 6

27. 12

29. 72

31. 56

33. 4

35. 40

37. 13

39. 6

41. 2,320

43. 45

45. 8 cm

47. 7 cm

49. 14

51. 2,590

53. 4

55. 7

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