6c Pythagorean Theorem

The Pythagorean Theorem

An angle that measures 90 degrees is called a *right angle*. If one of the angles of a triangle is a right angle, then the triangle is called a *right triangle*. It is traditional to mark the right angle with a little square (see Figure 6.1).

![Figure 6.1: Right triangle \( \triangle ABC \) has a right angle at vertex \( C \).](image)

**Right triangle terminology.**

- The longest side of the right triangle, the side directly opposite the right angle, is called the *hypotenuse* of the right triangle.

- The remaining two sides of the right triangle are called the *legs* of the right triangle.

**Proof of the Pythagorean Theorem**

Each side of the square in Figure 6.2 has been divided into two segments, one of length \( a \), the other of length \( b \).

We can find the total area of the square by squaring any one of the sides of the square.

\[
A = (a + b)^2 \quad \text{Square a side to find area.}
\]

\[
A = a^2 + 2ab + b^2 \quad \text{Squaring a binomial pattern.}
\]

Thus, the total area of the square is \( A = a^2 + 2ab + b^2 \).

A second approach to finding the area of the square is to sum the areas of the geometric parts that make up the square. We have four congruent right triangles, shaded in light red, with base \( a \) and height \( b \). The area of each of these triangles is found by taking one-half times the base times the height; i.e., the area of each triangles is \( (1/2)ab \). In the interior, we have a smaller square
with side $c$. Its area is found by squaring its side; i.e., the area of the smaller square is $c^2$.

The total area of the square is the sum of its parts, one smaller square and four congruent triangles. That is:

$$A = c^2 + 4 \left( \frac{1}{2}ab \right)$$

Adding the area of the interior square and the area of four right triangles.

$$A = c^2 + 2ab$$

Simplify: $4(1/2)ab = 2ab$.

The two expressions, $a^2 + 2ab + b^2$ and $c^2 + 2ab$, both represent the total area of the large square. Hence, they must be equal to one another.

$$a^2 + 2ab + b^2 = c^2 + 2ab$$

Each side of this equation represents the area of the large square.

$$a^2 + b^2 = c^2$$

Subtract $2ab$ from both sides.

The last equation, $a^2 + b^2 = c^2$, is called the Pythagorean Theorem.

**The Pythagorean Theorem.** If $a$ and $b$ are the legs of a right triangle and $c$ is its hypotenuse, then:

$$a^2 + b^2 = c^2$$

We say “The sum of the squares of the legs of a right triangle equals the square of its hypotenuse.”

**Good hint.** Note that the hypotenuse sits by itself on one side of the equation $a^2 + b^2 = c^2$. The legs of the hypotenuse are on the other side.

Let’s put the Pythagorean Theorem to work.
EXAMPLE 1. Find the missing side of the right triangle shown below.

Solution: First, write out the Pythagorean Theorem, then substitute the given values in the appropriate places.

\[
\begin{align*}
    a^2 + b^2 &= c^2 & \text{Pythagorean Theorem.} \\
    (4)^2 + (3^2) &= c^2 & \text{Substitute: 4 for } a, 3 \text{ for } b. \\
    16 + 9 &= c^2 & \text{Square: } 4^2 = 16, 3^2 = 9. \\
    25 &= c^2 & \text{Add: } 16 + 9 = 25. \\
\end{align*}
\]

The equation \( c^2 = 25 \) has two real solutions, \( c = -5 \) and \( c = 5 \). However, in this situation, \( c \) represents the length of the hypotenuse and must be a positive number. Hence:

\[
c = 5
\]

Nonnegative square root.

Thus, the length of the hypotenuse is 5.

EXAMPLE 2. An isosceles right triangle has a hypotenuse of length 8. Find the lengths of the legs.

Solution: In general, an isosceles triangle is a triangle with two equal sides. In this case, an isosceles right triangle has two equal legs. We’ll let \( x \) represent the length of each leg.
Use the Pythagorean Theorem, substituting $x$ for each leg and 8 for the hypotenuse.

\[ a^2 + b^2 = c^2 \quad \text{Pythagorean Theorem.} \]
\[ x^2 + x^2 = 8^2 \quad \text{Substitute: } x \text{ for } a, \ x \text{ for } b, \ 8 \text{ for } c. \]
\[ 2x^2 = 64 \quad \text{Combine like terms: } x + x = 2x. \]
\[ x^2 = 32 \quad \text{Divide both sides by 2.} \]

The equation \( x^2 = 32 \) has two real solutions, \( x = -\sqrt{32} \) and \( x = \sqrt{32} \). However, in this situation, \( x \) represents the length of each leg and must be a positive number. Hence:

\[ x = \sqrt{32} \quad \text{Nonnegative square root.} \]

Remember, your final answer must be in simple form. We must factor out a perfect square when possible.

\[ x = \sqrt{16}\sqrt{2} \quad \text{Factor out a perfect square.} \]
\[ x = 4\sqrt{2} \quad \text{Simplify: } \sqrt{16} = 4. \]

Thus, the length of each leg is \( 4\sqrt{2} \).

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**Applications**

**EXAMPLE 3.** A ladder 20 feet long leans against the garage wall. If the base of the ladder is 8 feet from the garage wall, how high up the garage wall does the ladder reach? Find an exact answer, then use your calculator to round your answer to the nearest tenth of a foot.

**Solution:** As always, we obey the Requirements for Word Problem Solutions.

1. **Set up a variable dictionary.** We’ll create a well-marked diagram for this purpose, letting \( h \) represent the distance between the base of the garage wall and the upper tip of the ladder.
2. Set up an equation. Using the Pythagorean Theorem, we can write:

\[8^2 + h^2 = 20^2\]  
Pythagorean Theorem.  
\[64 + h^2 = 400\]  
Square: \(8^2 = 64\) and \(20^2 = 400\).

3. Solve the equation.

\[h^2 = 336\]  
Subtract 64 from both sides.  
\[h = \sqrt{336}\]  
\(h\) will be the nonnegative square root.  
\[h = \sqrt{16\sqrt{21}}\]  
Factor out a perfect square.  
\[h = 4\sqrt{21}\]  
Simplify: \(\sqrt{16} = 4\).

4. Answer the question. The ladder reaches \(4\sqrt{21}\) feet up the wall.

5. Look back.
Using the Pythagorean Theorem:

\[ 8^2 + (4\sqrt{21})^2 = 20^2 \]
\[ 8^2 + (4)^2 (\sqrt{21})^2 = 20^2 \]
\[ 64 + (16)(21) = 400 \]
\[ 64 + 336 = 400 \]
\[ 400 = 400 \]