1.2 Solving Equations

In this section, we review the equation-solving skills that are prerequisite for successful completion of the material in this text. Before we list the tools used in the equation-solving process, let’s make sure that we understand what is meant by the phrase “solve for $x$.”

**Solve for $x$.** Using the properties that we provide, you must “isolate $x$,“ so that your final solution takes the form

$$x = \text{“Stuff,”}$$

where “Stuff” can be an expression containing numbers, constants, other variables, and mathematical operators such as addition, subtraction, multiplication, division, square root, and the like.

“Stuff” can even contain other mathematical functions, such as exponentials, logarithms, or trigonometric functions. However, it is essential that you understand that there is one thing “Stuff” must not contain, and that is the variable you are solving for, in this case, $x$. So, in a sense, you want to isolate $x$ on one side of the equation, and put all the other “Stuff” on the other side of the equation.

Now, let’s provide the tools to help you with this task.

**Property 1.** Let $a$ and $b$ be any numbers such that $a = b$. Then, if $c$ is any number,

$$a + c = b + c,$$

and,

$$a - c = b - c.$$

In words, the first of these tools allows us to add the same quantity to both sides of an equation without affecting equality. The second statement tells us that we can subtract the same quantity from both sides of an equation and still have equality.

Let’s look at an example.

**Example 2.** Solve the equation $x + 5 = 7$ for $x$.

The goal is to “isolate $x$ on one side of the equation. To that end, let’s subtract 5 from both sides of the equation, then simplify.

\[x = \text{“Stuff,”}\]

\[x = 7 - 5 = 2\]

\[x = 2\]
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\[
x + 5 = 7 \\
x + 5 - 5 = 7 - 5 \\
x = 2.
\]

It is important to check your solution by showing that \( x = 2 \) “satisfies” the original equation. To that end, substitute \( x = 2 \) in the original equation and simplify both sides of the result.

\[
x + 5 = 7 \\
2 + 5 = 7 \\
7 = 7
\]

This last statement (i.e., \( 7 = 7 \)) is a true statement, so \( x = 2 \) is a solution of the equation \( x + 5 = 7 \).

An important concept is the idea of equivalent equations.

**Equivalent Equations.** Two equations are said to be equivalent if and only if they have the same solution set. That is, two equations are equivalent if each of the solutions of the first equation is also a solution of the second equation, and vice-versa.

Thus, in **Example 2**, the equations \( x + 5 = 7 \) and \( x = 2 \) are equivalent, because they both have the same solution set \( \{2\} \). It is no coincidence that the tools in **Property 1** produce equivalent equations. Whenever you add the same amount to both sides of an equation, the resulting equation is equivalent to the original equation (they have the same solution set). This is also true for subtraction. When you subtract the same amount from both sides of an equation, the resulting equation has the same solutions as the original equation.

Let’s look at another example.

**Example 3.** Solve the equation \( x - 7 = 12 \) for \( x \).

We want to “isolate \( x \)” on one side of the equation, so we add 7 to both sides of the equation and simplify.

\[
x - 7 = 12 \\
x - 7 + 7 = 12 + 7 \\
x = 19
\]

We will leave it to our readers to check that \( x = 19 \) is a solution of \( x - 7 = 12 \).

Let’s pause for a moment and define what is meant by a monomial.
Definition 4. A monomial is an algebraic expression that is the product of a number and zero or more variables, each raised to some arbitrary exponent.

Examples of monomials are:

\[ 3x^2, \quad \text{or} \quad -4ab^2, \quad \text{or} \quad 25x^3y^5, \quad \text{or} \quad 17, \quad \text{or} \quad -11x. \]

Monomials are commonly referred to as “terms.” We often use algebraic expressions that are the sum of two or more terms. For example, the expression

\[ 3x^3 + 2x^2 - 7x + 8 \quad \text{or equivalently} \quad 3x^3 + 2x^2 + (-7x) + 8, \]

is the sum of four terms, namely, \( 3x^3 \), \( 2x^2 \), \(-7x \), and \( 8 \). Note that the terms are those parts of the expression that are separated by addition symbols.

Some mathematicians prefer use the word “term” in a more relaxed manner, simply stating that the terms of an algebraic expression are those components of the expression that are separated by addition symbols. For example, the terms of the expression

\[ 3x^2 - \frac{1}{x} + \frac{2x^2}{x + 3} \quad \text{or equivalently} \quad 3x^2 + \left( -\frac{1}{x} \right) + \frac{2x^2}{x + 3}, \]

are \( 3x^2 \), \(-1/x \), and \( 2x^2/(x + 3) \). This is the meaning we will use in this text.

Having made the definition of what is meant by a “term,” let’s return to our discussion of solving equations.

Example 5. Solve the equation \( 3x - 3 = 2x + 4 \) for \( x \).

We will isolate all terms containing an \( x \) on the left side of this equation (we could just as well have chosen to isolate terms containing \( x \) on the right side of the equation). To this end, we don’t want the \(-3\) on the left side of the equation (we want it on the right), so we add 3 to both sides of the equation and simplify.

\[
3x - 3 = 2x + 4 \\
3x - 3 + 3 = 2x + 4 + 3 \\
3x = 2x + 7
\]

Remember that we have chosen to isolate all terms containing \( x \) on the left side of the equation. So, for our next step, we choose to subtract \( 2x \) from both sides of the equation (this will “move” it from the right over to the left), then simplify.

\[
3x = 2x + 7 \\
3x - 2x = 2x + 7 - 2x \\
x = 7
\]

To check the solution, substitute \( x = 7 \) in the original equation to obtain...
\[ 3x - 3 = 2x + 4 \]
\[ 3(7) - 3 = 2(7) + 4 \]
\[ 21 - 3 = 14 + 4 \]
\[ 18 = 18 \]

The last line is a true statement, so \( x = 7 \) checks and is a solution of \( 3x - 3 = 2x + 4 \).

If you use the technique of Example 5 repeatedly, there comes a point when you tire of showing the addition or subtraction of the same amount on both sides of your equation. Here is a tool, which, if carefully used, will greatly simplify your work.²

**Useful Shortcut.** When you move a term from one side of an equation to the other, that is, when you move a term from one side of the equal sign to the other side, simply change its sign.

Let’s see how we would apply this shortcut to the equation of Example 5. Start with the original equation,

\[ 3x - 3 = 2x + 4, \]

then move all terms containing an \( x \) to the left side of the equation, and move all other terms to the right side of the equation. Remember to change the sign of a term if it moves from one side of the equals sign to the other. If a term does not move from one side of the equation to the other, leave its sign alone. The result would be

\[ 3x - 2x = 4 + 3. \]

Thus, \( x = 7 \) and you are finished.

It is important to note that when we move the \(-3\) from the left-hand side of the above equation to the right-hand side of the equation and change its sign, what we are actually doing is adding 3 to both sides of the equation. A similar statement explains that moving \( 2x \) from the right-hand side to the left-hand side and changing its sign is simply a shortcut for subtracting \( 2x \) from both sides of the equation.

Here are two more useful tools for solving equations.

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² You should be aware that mathematics educators seemingly divide into two distinct camps regarding this tool: some refuse to let their students use it, others are comfortable with their students using it. There are good reasons for this dichotomy which we won’t go into here, but you should check to see how your teacher feels about your use of this tool in your work.
Property 6. Let $a$ and $b$ be any numbers such that $a = b$. Then, if $c$ is any number other than zero,

$$ac = bc.$$ 

If $c$ is any number other than zero, then

$$\frac{a}{c} = \frac{b}{c}.$$ 

In words, the first of these tools allows us to multiply both sides of an equation by the same number. A similar statement holds for division, provided we do not divide by zero (division by zero is meaningless). Both of these tools produce equivalent equations.

Let’s look at an example.

Example 7. Solve the equation $5x = 15$ for $x$.

In this case, only one term contains the variable $x$ and this term is already isolated on one side of the equation. We will divide both sides of this equation by 5, then simplify,

$$\begin{align*}
5x &= 15 \\
\frac{5x}{5} &= \frac{15}{5} \\
x &= 3.
\end{align*}$$

We’ll leave it to our readers to check this solution.

Example 8. Solve the equation $x/2 = 7$ for $x$.

Again, there is only one term containing $x$ and it is already isolated on one side of the equation. We will multiply both sides of the equation by 2, then simplify,

$$\begin{align*}
\frac{x}{2} &= 7 \\
2 \left( \frac{x}{2} \right) &= 2(7) \\
x &= 14.
\end{align*}$$

Again, we will leave it to our readers to check this solution.

Let’s apply everything we’ve learned in the next example.

Example 9. Solve the equation $7x - 4 = 5 - 3x$ for $x$.

Note that we have terms containing $x$ on both sides of the equation. Thus, the first step is to isolate the terms containing $x$ on one side of the equation (left or right,
We will move the terms containing \( x \) to the left side of the equation, everything else will be moved to the right side of the equation. Remember the rule, if a term moves from one side of the equal sign to the other, change the sign of the term you are moving. Thus,

\[
7x - 4 = 5 - 3x \\
7x + 3x = 5 + 4.
\]

Simplify.

\[
10x = 9
\]

Divide both sides of this last result by 10.

\[
\frac{10x}{10} = \frac{9}{10} \quad \Rightarrow \quad x = \frac{9}{10}
\]

To check this solution, substitute \( x = \frac{9}{10} \) into both sides of the original equation and simplify.

\[
7\left(\frac{9}{10}\right) - 4 = 5 - 3\left(\frac{9}{10}\right)
\]

\[
\frac{63}{10} - 4 = 5 - \frac{27}{10}
\]

We’ll need a common denominator to verify that our solution is correct. That is,

\[
\frac{63}{10} - \frac{40}{10} = \frac{50}{10} - \frac{27}{10}
\]

\[
\frac{23}{10} = \frac{23}{10}.
\]

Thus, \( x = \frac{9}{10} \) checks and is a solution of \( 7x - 4 = 5 - 3x \).

Note that the check can sometimes be more difficult than solving the equation. This is one of the reasons that we tend to get lazy and not check our solutions. However, we shouldn’t need to tell you what will probably happen if you do not check your work.

There is a workaround that involves the use of the graphing calculator. We first store the solution for \( x \) in our calculator, then calculate each side of the original equation and compare results.

1. Enter \( \frac{9}{10} \) in your calculator window, then
2. push the \( \text{STO} \) key, then
3. push the \( x \) key followed by the \( \text{ENTER} \) key.

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\[3\] Although moving all the terms containing \( x \) to the right side is alright, it is often preferable to have the \( x \) terms on the left side of the equation in order to end up with \( x = \text{“Stuff.”} \)
The result is shown in Figure 1(a).

Now that we’ve stored \( x = \frac{9}{10} \) in the calculator’s memory, let’s evaluate each side of the equation \( 7x - 4 = 5 - 3x \) at this value of \( x \). Enter \( 7 \times x - 4 \) in your calculator and press ENTER. The result is shown in Figure 1(b), where we see that \( 7x - 4 \), evaluated at \( x = \frac{9}{10} \), equals 2.3.

Next, enter \( 5 - 3 \times x \) and press ENTER. The result is shown in Figure 1(c), where we see that \( 5 - 3x \), evaluated at \( x = \frac{9}{10} \), also equals 2.3 (by the way, this is equivalent to the \( \frac{23}{10} \) we found in our hand check above).

Because the expressions on each side of the equation are equal when \( x = \frac{9}{10} \) (both equal 2.3), the solution checks.

![Figure 1. Checking the solution of \( 7x - 4 = 5 - 3x \) with the graphing calculator.](image)

If you need to solve an equation that contains fractions, one very useful strategy is to clear the equations of fractions by multiplying both sides of the equation by the least common denominator.

**Example 10.** Solve the equation

\[
\frac{2}{3} x - \frac{3}{4} = \frac{1}{4} - \frac{3}{2} x
\]

for \( x \).

The least common denominator is 12, so we multiply both sides of this equation by 12.

\[
12 \left( \frac{2}{3} x - \frac{3}{4} \right) = 12 \left( \frac{1}{4} - \frac{3}{2} x \right)
\]

Distribute the 12 and simplify.

\[
12 \left( \frac{2}{3} \right) x - 12 \left( \frac{3}{4} \right) = 12 \left( \frac{1}{4} \right) - 12 \left( \frac{3}{2} \right) x
\]

\[
8x - 9 = 3 - 18x
\]

Move all terms containing \( x \) to the left side of the equation, everything else to the right, then simplify.
\[8x + 18x = 3 + 9\]
\[26x = 12\]
Divide both sides of this last result by 26 and simplify (always reduce to lowest terms — in this case we can divide both numerator and denominator by 2).
\[
\frac{26x}{26} = \frac{12}{26}
\]
\[x = \frac{6}{13}\]
We leave it to our readers to check this solution. Use your graphing calculator as demonstrated in Example 9.

You can clear decimals from an equation by multiplying by the appropriate power of 10.\(^4\)

\textbf{Example 11.} Solve the equation \(1.23x - 5.46 = 3.72x\) for \(x\).

Let’s multiply both sides of this equation by 100, which moves the decimal two places to the right, which is enough to clear the decimals from this problem.
\[100(1.23x - 5.46) = 100(3.72x)\]

Distribute and simplify.
\[100(1.23x) - 100(5.46) = 100(3.72x)\]
\[123x - 546 = 372x\]
Move each term containing an \(x\) to the right side of the equation (the first time we’ve chosen to do this — it avoids a negative sign in the coefficient of \(x\)) and simplify.
\[-546 = 372x - 123x\]
\[-546 = 249x\]
Divide both sides of the equation by 249 and simplify (in this case we can reduce, dividing numerator and denominator by 3).
\[
\frac{-546}{249} = \frac{249x}{249}
\]
\[
-\frac{182}{83} = x
\]
Rewrite your answer, placing \(x\) on the left side of the equation.
\[x = -\frac{182}{83}\]
Check your result with your calculator. It is important to be sure that you always use the original problem when you check your result. The steps are shown in Figure 2(a), (b), and (c).

\(^4\) Multiplying by 10 moves the decimal one place to the right, multiplying by 100 moves the decimal two places to the right, etc.
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Figure 2. Checking that $x = -182/83$ is a solution of $1.23x - 5.46 = 3.72x$.

**Formulae**

Science is filled with formulae that involve more than one variable and a number of constants. In chemistry and physics, the instructor will expect that you can manipulate these equations, solving for one variable or constant in terms of the others in the equation.

There is nothing new to say here, as you should follow the same rules that we’ve given heretofore when the only variable was $x$. However, students usually find these a bit intimidating because of the presence of multiple variables and constants, so let’s take our time and walk through a couple of examples.

**Example 12.** Isaac Newton is credited with the formula that determines the magnitude $F$ of the force of attraction between two planets. The formula is

$$F = \frac{GmM}{r^2},$$

where $m$ is the mass of the smaller planet, $M$ is the mass of the larger planet, $r$ is the distance between the two planets, and $G$ is a universal gravitational constant. Solve this equation for $G$.

First, a word of caution.

**Warning 13.** When using formulae of science, never change the case of a variable or constant. If it is uppercase, write it in uppercase on your homework. The same directive applies if the variable or constant is presented in lowercase. Write it in lowercase on your homework.

This equation has fractions in it, so we will begin by multiplying both sides of the equation by the common denominator, which in this case is $r^2$.

$$r^2(F) = r^2 \left( \frac{GmM}{r^2} \right)$$

This gives us
\[ r^2 F = G m M. \]

In this case, there is only one term with \( G \), and that term is already isolated on one side of the equation. The next step is to divide both sides of the equation by the coefficient of \( G \), then simplify.

\[ \frac{r^2 F}{mM} = \frac{G m M}{mM} \]

\[ \frac{r^2 F}{mM} = G \]

Hence,

\[ G = \frac{r^2 F}{mM}. \]

Note that we have \( G = \text{“Stuff”} \), and most importantly, the “Stuff” has no occurrence of the variable \( G \). This is what it means to “solve for \( G \).”

Let’s look at a final example.

**Example 14.** Water freezes at 0° Celsius and boils at 100° Celsius. Americans are probably more familiar with Fahrenheit temperature, where water freezes at 32° Fahrenheit and boils at 212° Fahrenheit. The formula to convert Celsius temperature \( C \) into Fahrenheit temperature \( F \) is

\[ F = \frac{9}{5} C + 32. \]

Solve this equation for \( C \).

Once again, the equation has fractions in it, so our first move will be to eliminate the fractions by multiplying both sides of the equation by the common denominator (5 in this case).

\[ 5F = 5 \left( \frac{9}{5} C + 32 \right) \]

\[ 5F = 9C + 160 \]

We’re solving for \( C \), so move all terms containing a \( C \) to one side of the equation, and all other terms to the other side of the equation.

\[ 5F - 160 = 9C \]

Divide both sides of this last equation by 9.
\[
\begin{align*}
\frac{5F - 160}{9} &= \frac{9C}{9} \\
\frac{5F - 160}{9} &= C
\end{align*}
\]

Thus,

\[C = \frac{5F - 160}{9}.\]

Note that we have \(C = \text{“Stuff,”} \) and most importantly, the “Stuff” has no occurrence of the variable \(C\). This is what it means to solve for \(C\).

Once you’ve solved a formula from science for a particular variable, you can use the result to make conversions or predictions.

**Example 15.** In Example 14, the relationship between Fahrenheit and Celsius temperatures is given by the result

\[C = \frac{5F - 160}{9}.\]  

(16)

Above the bank in Eureka, California, a sign proclaims that the Fahrenheit temperature is \(40^\circ F\). What is the Celsius temperature?

Substitute the Fahrenheit temperature into formula (16). That is, substitute \(F = 40\).

\[C = \frac{5F - 160}{9} = \frac{5(40) - 160}{9} = \frac{40}{9} \approx 4.44\]

Hence, the Celsius temperature is approximately \(4.44^\circ C\). Note that you should always include units with your final answer.