7.5 Sums and Differences of Rational Functions

In this section we concentrate on finding sums and differences of rational expressions. However, before we begin, we need to review some fundamental ideas and technique.

First and foremost is the concept of the multiple of an integer. This is best explained with a simple example. The multiples of 8 is the set of integers \(\{8k : k \text{ is an integer}\}\). In other words, if you multiply 8 by 0, ±1, ±2, ±3, ±4, etc., you produce what is known as the multiples of 8.

Multiples of 8 are: 0, ±8, ±16, ±24, ±32, etc.

However, for our purposes, only the positive multiples are of interest. So we will say:

Multiples of 8 are: 8, 16, 24, 32, 40, 48, 56, 64, 72, . . .

Similarly, we can list the positive multiples of 6.

Multiples of 6 are: 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, . . .

We’ve framed those numbers that are multiples of both 8 and 6. These are called the common multiples of 8 and 6.

Common multiples of 8 and 6 are: 24, 48, 72, . . .

The smallest of this list of common multiples of 8 and 6 is called the least common multiple of 8 and 6. We will use the following notation to represent the least common multiple of 8 and 6: \(\text{LCM}(8, 6)\).

Hopefully, you will now feel comfortable with the following definition.

**Definition 1.** Let \(a\) and \(b\) be integers. The least common multiple of \(a\) and \(b\), denoted \(\text{LCM}(a, b)\), is the smallest positive multiple that \(a\) and \(b\) have in common.

For larger numbers, listing multiples until you find one in common can be impractical and time consuming. Let’s find the least common multiple of 8 and 6 a second time, only this time let’s use a different technique.

First, write each number as a product of primes in exponential form.

\[
8 = 2^3 \\
6 = 2 \cdot 3
\]
Here’s the rule.

**A Procedure to Find the LCM.** To find the LCM of two integers, proceed as follows.

1. Express the prime factorization of each integer in exponential format.
2. To find the least common multiple, write down every prime number that appears, then affix the largest exponent of that prime that appears.

In our example, the primes that occur are 2 and 3. The highest power of 2 that occurs is $2^3$. The highest power of 3 that occurs is $3^1$. Thus, the LCM$(8, 6)$ is

$$\text{LCM}(8, 6) = 2^3 \cdot 3^1 = 24.$$  

Note that this result is identical to the result found above by listing all common multiples and choosing the smallest.

Let’s try a harder example.

**Example 2.** Find the least common multiple of 24 and 36.

Using the first technique, we list the multiples of each number, framing the multiples in common.

- Multiples of 24: 24, 48, 72, 96, 120, 144, ... $168, ...$
- Multiples of 36: 36, 72, 108, 144, 180, ...$216, ...$

The multiples in common are 72, 144, etc., and the least common multiple is LCM$(24, 36) = 72$.

Now, let’s use our second technique to find the least common multiple (LCM). First, express each number as a product of primes in exponential format.

$$24 = 2^3 \cdot 3$$
$$36 = 2^2 \cdot 3^2$$

To find the least common multiple, write down every prime that occurs and affix the highest power of that prime that occurs. Thus, the highest power of 2 that occurs is $2^3$, and the highest power of 3 that occurs is $3^2$. Thus, the least common multiple is

$$\text{LCM}(24, 36) = 2^3 \cdot 3^2 = 8 \cdot 9 = 72.$$
Addition and Subtraction Defined

Imagine a pizza that has been cut into 12 equal slices. Then, each slice of pizza represents 1/12 of the entire pizza.

If Jimmy eats 3 slices, then he has consumed 3/12 of the entire pizza. If Margaret eats 2 slices, then she has consumed 2/12 of the entire pizza. It’s clear that together they have consumed

$$\frac{3}{12} + \frac{2}{12} = \frac{5}{12}$$

of the pizza. It would seem that adding two fractions with a common denominator is as simple as eating pizza! Hopefully, the following definition will seem reasonable.

**Definition 3.** To add two fractions with a common denominator, such as \( \frac{a}{c} \) and \( \frac{b}{c} \), add the numerators and divide by the common denominator. In symbols,

$$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}.$$ 

Note how this definition agrees precisely with our pizza consumption discussed above. Here are some examples of adding fractions having common denominators.

$$\frac{5}{21} + \frac{3}{21} = \frac{5 + 3}{21} = \frac{8}{21}$$

$$\frac{2}{x+2} + \frac{x-3}{x+2} = \frac{2 + (x-3)}{x+2} = \frac{2 + x - 3}{x+2} = \frac{x - 1}{x+2}$$

Subtraction works in much the same way as does addition.

**Definition 4.** To subtract two fractions with a common denominator, such as \( \frac{a}{c} \) and \( \frac{b}{c} \), subtract the numerators and divide by the common denominator. In symbols,

$$\frac{a}{c} - \frac{b}{c} = \frac{a - b}{c}.$$ 

Here are some examples of subtracting fractions already having common denominators.

$$\frac{5}{21} - \frac{3}{21} = \frac{5 - 3}{21} = \frac{2}{21}$$

$$\frac{2}{x+2} - \frac{x-3}{x+2} = \frac{2 - (x-3)}{x+2} = \frac{2 - x + 3}{x+2} = \frac{5 - x}{x+2}$$
In the example on the right, note that it is extremely important to use grouping symbols when subtracting numerators. Note that the minus sign in front of the parenthetical expression changes the sign of each term inside the parentheses.

There are times when a sign change will provide a common denominator.

Example 5. Simplify

\[
\frac{x}{x - 3} - \frac{2}{3 - x}. \tag{6}
\]

State all restrictions.

At first glance, it appears that we do not have a common denominator. On second glance, if we make a sign change on the second fraction, it might help. So, on the second fraction, let’s negate the denominator and fraction bar to obtain

\[
\frac{x}{x - 3} - \frac{2}{3 - x} = \frac{x}{x - 3} + \frac{2}{x - 3} = \frac{x + 2}{x - 3}. \tag{7}
\]

The denominators \(x - 3\) or \(3 - x\) are zero when \(x = 3\). Hence, 3 is a restricted value. For all other values of \(x\), the left-hand side of

\[
\frac{x}{x - 3} - \frac{2}{3 - x} = \frac{x + 2}{x - 3}. \tag{7}
\]

is identical to the right-hand side.

This is easily tested using the table utility on the graphing calculator, as shown in the sequence of screenshots in Figure 1. First load the left- and right-hand sides of equation (7) into \(Y_1\) and \(Y_2\) in the \(Y=\) menu of your graphing calculator, as shown in Figure 1(a). Press 2nd TBLSET and make the changes shown in Figure 1(b). Press 2nd TABLE to produce the table shown in Figure 1(c). Note the ERR (error) message at the restriction \(x = 3\), but note also the agreement of \(Y_1\) and \(Y_2\) for all other values of \(x\).

![Figure 1](image_url)

**Figure 1.** Using the table feature of the graphing calculator to check the result in equation (7).
**Equivalent Fractions**

If you slice a pizza into four equal pieces, then consume two of the four slices, you’ve consumed half of the pizza. This motivates the fact that

\[
\frac{1}{2} = \frac{2}{4}.
\]

Indeed, if you slice the pizza into six equal pieces, then consume three slices, you’ve consumed half of the pizza, so it’s fair to say that \( \frac{3}{6} = \frac{1}{2} \). Indeed, all of the following fractions are equivalent:

\[
\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{6}{12} = \frac{7}{14} = \cdots
\]

A more formal way to demonstrate that \( \frac{1}{2} \) and \( \frac{7}{14} \) are equal is to start with the fact that \( \frac{1}{2} = \frac{1}{2} \times 1 \), then replace \( 1 \) with \( \frac{7}{7} \) and multiply.

\[
\frac{1}{2} = \frac{1}{2} \times 1 = \frac{1}{2} \times \frac{7}{7} = \frac{7}{14}
\]

Here’s another example of this principle in action, only this time we replace \( 1 \) with \( \frac{(x-2)}{(x-2)} \).

\[
\frac{3}{x+2} = \frac{3}{x+2} \cdot 1 = \frac{3}{x+2} \cdot \frac{x-2}{x-2} = \frac{3(x-2)}{(x+2)(x-2)}
\]

In the next example we replace \( 1 \) with \( \frac{(x(x-3))}{(x-3)} \).

\[
\frac{2}{x-4} = \frac{2}{x-4} \cdot 1 = \frac{2}{x-4} \cdot \frac{x(x-3)}{x(x-3)} = \frac{2x(x-3)}{x(x-4)(x-3)}
\]

Now, let’s apply the concept of equivalent fractions to add and subtract fractions with different denominators.

**Adding and Subtracting Fractions with Different Denominators**

In this section we show our readers how to add and subtract fractions having different denominators. For example, suppose we are asked to add the following fractions.

\[
\frac{5}{12} + \frac{5}{18}
\]

First, we must find a “common denominator.” Fortunately, the machinery to find the “common denominator” is already in place. It turns out that the least common denominator for 12 and 18 is the least common multiple of 12 and 18.

\[
\begin{align*}
18 &= 2 \cdot 3^2 \\
12 &= 2^2 \cdot 3 \\
\text{LCD}(12, 18) &= 2^2 \cdot 3^2 = 36
\end{align*}
\]
The next step is to create equivalent fractions using the LCD as the denominator. So, in the case of $\frac{5}{12}$,

$$\frac{5}{12} = \frac{5}{12} \cdot 1 = \frac{5}{12} \cdot \frac{3}{3} = \frac{15}{36}.$$ 

In the case of $\frac{5}{18}$,

$$\frac{5}{18} = \frac{5}{18} \cdot 1 = \frac{5}{18} \cdot \frac{2}{2} = \frac{10}{36}.$$ 

If we replace the fractions in equation (8) with their equivalent fractions, we can then add the numerators and divide by the common denominator, as in

$$\frac{5}{12} + \frac{5}{18} = \frac{15}{36} + \frac{10}{36} = \frac{25}{36}.$$ 

Let’s examine a method of organizing the work that is more compact. Consider the following arrangement, where we’ve used color to highlight the form of 1 required to convert the fractions to equivalent fractions with a common denominator of 36.

$$\frac{5}{12} + \frac{5}{18} = \frac{5}{12} \cdot \frac{3}{3} + \frac{5}{18} \cdot \frac{2}{2} = \frac{15}{36} + \frac{10}{36} = \frac{25}{36}.$$ 

Let’s look at a more complicated example.

**Example 9.** Simplify the expression

$$\frac{x + 3}{x + 2} - \frac{x + 2}{x + 3}.$$ 

State all restrictions.

The denominators are already factored. If we take each factor that appears to the highest exponential power that appears, our least common denominator is $(x+2)(x+3)$. Our first task is to make equivalent fractions having this common denominator.

$$\frac{x + 3}{x + 2} - \frac{x + 2}{x + 3} = \frac{x + 3}{x + 2} \cdot \frac{x + 3}{x + 3} - \frac{x + 2}{x + 3} \cdot \frac{x + 2}{x + 2} = \frac{x^2 + 6x + 9}{(x + 2)(x + 3)} - \frac{x^2 + 4x + 4}{(x + 2)(x + 3)}.$$ 

Now, subtract the numerators and divide by the common denominator.
\[
\frac{x + 3}{x + 2} - \frac{x + 2}{x + 3} = \frac{(x^2 + 6x + 9) - (x^2 + 4x + 4)}{(x + 2)(x + 3)}
\]
\[
= \frac{x^2 + 6x + 9 - x^2 - 4x - 4}{(x + 2)(x + 3)}
\]
\[
= \frac{2x + 5}{(x + 2)(x + 3)}
\]

Note the use of parentheses when we subtracted the numerators. Note further how the minus sign negates each term in the parenthetical expression that follows the minus sign.

**Tip 11.** Always use grouping symbols when subtracting the numerators of fractions.

In the final answer, the factors \(x + 2\) and \(x + 3\) in the denominator are zero when \(x = -2\) or \(x = -3\). These are the restrictions. No other denominators, in the original problem or in the body of our work, provide additional restrictions.

Thus, for all values of \(x\), except the restricted values \(-2\) and \(-3\), the left-hand side of

\[
\frac{x + 3}{x + 2} - \frac{x + 2}{x + 3} = \frac{2x + 5}{(x + 2)(x + 3)} \quad (12)
\]

is identical to the right-hand side. This claim is easily tested on the graphing calculator which is evidenced in the sequence of screen captures in Figure 2. Note the ERR (error) message at each restricted value of \(x\) in Figure 2(c), but also note the agreement of \(Y1\) and \(Y2\) for all other values of \(x\).

(a) (b) (c)

**Figure 2.** Using the table feature of the graphing calculator to check the result in equation (12).
Let’s look at another example.

**Example 13. Simplify the expression**

\[
\frac{4}{x^2 + 6x + 5} - \frac{2}{x^2 + 8x + 15}.
\]

*State all restrictions.*

First, factor each denominator.

\[
\frac{4}{x^2 + 6x + 5} - \frac{2}{x^2 + 8x + 15} = \frac{4}{(x + 1)(x + 5)} - \frac{2}{(x + 3)(x + 5)}.
\]

The least common denominator, or least common multiple (LCM), requires that we write down each factor that occurs, then affix the highest power of that factor that occurs. Because all factors in the denominators are raised to an understood power of one, the LCD (least common denominator) or LCM is \((x + 1)(x + 5)(x + 3)\).

Next, we make equivalent fractions having this common denominator.

\[
\frac{4}{x^2 + 6x + 5} - \frac{2}{x^2 + 8x + 15} = \frac{4}{(x + 1)(x + 5)} \cdot \frac{x + 3}{x + 3} - \frac{2}{(x + 3)(x + 5)} \cdot \frac{x + 1}{x + 1}.
\]

Subtract the numerators and divide by the common denominator. Be sure to use grouping symbols, particularly with the minus sign that is in play.

\[
\frac{4}{x^2 + 6x + 5} - \frac{2}{x^2 + 8x + 15} = \frac{(4x + 12) - (2x + 2)}{(x + 3)(x + 5)(x + 1)}.
\]

\[
= \frac{4x + 12 - 2x - 2}{(x + 3)(x + 5)(x + 1)}.
\]

\[
= \frac{2x + 10}{(x + 3)(x + 5)(x + 1)}.
\]

Finally, we should always make sure that our answer is reduced to lowest terms. With that thought in mind, we factor the numerator in hopes that we can get a common factor to cancel.

\[
\frac{4}{x^2 + 6x + 5} - \frac{2}{x^2 + 8x + 15} = \frac{2(x + 5)}{(x + 3)(x + 5)(x + 1)}.
\]

The denominators have factors of \(x + 3\), \(x + 5\) and \(x + 1\), so the restrictions are \(x = -3\), \(x = -5\), and \(x = -1\), respectively. For all other values of \(x\), the left-hand side of

\[
\frac{4}{x^2 + 6x + 5} - \frac{2}{x^2 + 8x + 15} = \frac{2}{(x + 3)(x + 1)}.
\]
is identical to its right-hand side. Again, this is easily tested using the table feature of the graphing calculator, as shown in the screenshots in Figure 3. Again, note the ERR (error) messages at each restricted value of \(x\), but also note that \(Y_1\) and \(Y_2\) agree for all other values of \(x\).

**Figure 3.** Using the table feature of the graphing calculator to check the result in equation (14).

Let’s look at another example.

**Example 15.** Simplify the expression

\[
\frac{x - 3}{x^2 - 1} + \frac{1}{x + 1} - \frac{1}{1 - x}.
\]

State all restrictions.

First, factor all denominators.

\[
\frac{x - 3}{x^2 - 1} + \frac{1}{x + 1} - \frac{1}{1 - x} = \frac{x - 3}{(x + 1)(x - 1)} + \frac{1}{x + 1} - \frac{1}{1 - x}.
\]

If we’re not careful, we might be tempted to take one of each factor and use \((x + 1)(x - 1)(1 - x)\) as a common denominator. However, let’s first make two negations of the last of the three fractions on the right, negating the fraction bar and denominator to get

\[
\frac{x - 3}{x^2 - 1} + \frac{1}{x + 1} - \frac{1}{1 - x} = \frac{x - 3}{(x + 1)(x - 1)} + \frac{1}{x + 1} + \frac{1}{x - 1}.
\]

Now we can see that a common denominator of \((x + 1)(x - 1)\) will suffice. Let’s make equivalent fractions with this common denominator.

\[
\frac{x - 3}{x^2 - 1} + \frac{1}{x + 1} - \frac{1}{1 - x} = \frac{x - 3}{(x + 1)(x - 1)} + \frac{1}{x + 1} + \frac{1}{x - 1}.
\]

Add the numerators and divide by the common denominator. Even though grouping symbols are not as critical in this problem (because of the plus signs), we still think it good practice to use them.
Finally, always make sure that your final answer is reduced to lowest terms. With that thought in mind, we factor the numerator in hopes that we can get a common factor to cancel.

\[
\frac{x - 3}{x^2 - 1} + \frac{1}{x + 1} - \frac{1}{1 - x} = \frac{(x - 3) + (x - 1) + (x + 1)}{(x + 1)(x - 1)}
\]

\[
= \frac{3x - 3}{(x + 1)(x - 1)}
\]

The factors \(x + 1\) and \(x - 1\) in the denominator produce restrictions \(x = -1\) and \(x = 1\), respectively. However, for all other values of \(x\), the left-hand side of

\[
\frac{x - 3}{x^2 - 1} + \frac{1}{x + 1} - \frac{1}{1 - x} = \frac{3}{x + 1}
\]

is identical to the right-hand side. Again, this is easily checked on the graphing calculator as shown in the sequence of screenshots in Figure 4.

Figure 4. Using the table feature of the graphing calculator to check the result in equation (16).

Again, note the ERR (error) messages at each restriction, but also note that the values of \(Y1\) and \(Y2\) agree for all other values of \(x\).
Let’s look at an example using function notation.

**Example 17.** If the function \( f \) and \( g \) are defined by the rules

\[
 f(x) = \frac{x}{x + 2} \quad \text{and} \quad g(x) = \frac{1}{x},
\]

simplify \( f(x) - g(x) \).

First,

\[
 f(x) - g(x) = \frac{x}{x + 2} - \frac{1}{x}.
\]

Note how tempting it would be to cancel. However, canceling would be an error in this situation, because subtraction requires a common denominator.

\[
 f(x) - g(x) = \frac{x}{x + 2} \cdot \frac{x}{x} - \frac{1}{x} \cdot \frac{x + 2}{x + 2}
\]

\[
 = \frac{x^2}{x(x + 2)} - \frac{x + 2}{x(x + 2)}
\]

Subtract numerators and divide by the common denominator. This requires that we “distribute” the minus sign.

\[
 f(x) - g(x) = \frac{x^2 - (x + 2)}{x(x + 2)}
\]

\[
 = \frac{x^2 - x - 2}{x(x + 2)}
\]

This result is valid for all values of \( x \) except 0 and \( -2 \). We leave it to our readers to verify that this result is reduced to lowest terms. You might want to check the result on your calculator as well.